Appendix A: Spectral and Geometric Contributions to a

A recurring theme in quantum field theory and geometric topology is the appearance of a mass gap: a nonzero separation between allowed states. In quantum field theory, a mass gap is spectral. It is a lower bound in the excitation spectrum of an operator: the lowest nontrivial excitation lies strictly above the vacuum state.

In the Yang-Mills context, the mass gap is a property of the quantum Hamiltonian. It says that the first nonzero excitation of the gauge field has positive mass. This is an analytic and spectral property, arising from the structure of the underlying quantum operator.

In contrast, hyperbolic geometry possesses a distinct kind of mass gap. For Lorentzian and hyperbolic manifolds, the two-sheeted hyperboloid

$$-{x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 = -1$$

exhibits an intrinsic separation between sheets—a geometric "gap" that distinguishes time-like and space-like sectors. When normalized by physical parameters (such as the Planck mass), this gap defines a fundamental separation scale between geometric states.

Thus, we have two distinct notions: a spectral mass gap, as in Yang-Mills theory, emerging from the analytic eigenvalue structure of a quantum operator; and a geometric gap, emerging from the topology and metric structure of a hyperbolic manifold and its Lorentzian embedding.

These notions are generally discussed independently. What follows is an argument that, in the context of the hyperbolic partition equation introduced in this work, they arise as two contributions to a single structural parameter.

The hyperbolic partition equation is written as:

$$\frac{1}{x} + x + \frac{x^3}{2\pi} = \left(i^i\right)^{-4\pi/8} - \frac{m_p}{\mathrm{kg}}.$$

Where $\pi =$ Archimedes' constant, $i =$ the imaginary unit, $m_p =$ the Planck mass, and $\mathrm{kg} =$ the kilogram.

The right-hand side defines a spectral parameter

$$a = \left(i^{i}\right)^{-4\pi/8} - \frac{m_p}{\mathrm{kg}},$$

for the associated quartic polynomial. The roots of this quartic, $ж_1$, $ж_2$, $ж_3$, $ж_4$, determine the entire decomposition structure used in the analysis of the constants of Nature: their products, sums, quadrances, and polar components. Thus, a functions as an eigenvalue-setting parameter: changing a changes the roots, and therefore changes the discrete modes of the partition structure.

The parameter a naturally decomposes into two pieces.

$\left(i^{i}\right)^{-4\pi/8}$ is a purely analytic object arising from complex iteration theory and encodes a form of hyperbolic analytic continuation. It is independent of any physical scale and is structurally analogous to a spectral contribution in quantum field theory—i.e., the part of the Hamiltonian spectrum determined by analytic properties alone.

m_p/kilogram is the normalized Planck mass. It is dimensionless and represents the geometric mass gap associated with the hyperbolic structure itself. Hence:

$$a = \left(i^{i}\right)^{-4\pi/8} - \frac{m_p}{\mathrm{kg}}.$$

spectral **geometric**

The analytic term $\left(i^{i}\right)^{-4\pi/8}$ is a pure dimensionless number arising from complex exponentiation and analytic continuation. It defines the "shape" of the spectrum independent of physical units, and determines the allowable modes of the associated quartic.

The geometric term m_p/kg plays the same role as the geometric hyperbolic mass gap: It defines a structural separation between hyperbolic sheets, encoding a topological/metric constraint of the underlying space and introducing a physical scale into what would otherwise be a purely analytic relation.

The parameter a joins these two different kinds of contribution: an analytic spectral-like term, and a normalized geometric mass term. The interaction between these two contributions is what determines the observed partition roots, including the first root $ж_1$, whose square is the fine-structure constant. This association suggests that the fine-structure constant itself emerges at the intersection of: the analytic spectral structure of the system, and the geometric mass-gap boundary associated with the Planck scale.

In a standard Yang-Mills theory, the mass gap is spectral: it arises from the operator’s eigenvalues. In contrast, hyperbolic geometry introduces a structural separation (the two-sheeted hyperboloid), which plays the role of an intrinsic geometric gap.

These two notions appear as complementary components of the same partition equation. The fine-structure constant—and the system of constants it participates in—then appears to be part of a single coherent partition geometry whose analytic and geometric contributions combine in the parameter a.

Appendix B: Rational Correspondences of Möbius Maps $\boldsymbol{F(t)}$ and $\boldsymbol{G(t)}$

$$F\left(\frac{2\pi}{1}\right) = -\frac{1}{2} \qquad G\left(-\frac{1}{2}\right) = \frac{2\pi}{1} \qquad F\left(-\frac{2\pi}{1}\right) \to \infty \qquad G\left(-\frac{1}{1}\right) = 0$$

$$F\left(\frac{2\pi}{2}\right) = -\frac{2}{3} \qquad G\left(-\frac{2}{3}\right) = \frac{2\pi}{2} \qquad F\left(-\frac{2\pi}{2}\right) = -\frac{2}{1} \qquad G\left(-\frac{2}{1}\right) = -\frac{2\pi}{2}$$

$$F\left(\frac{2\pi}{3}\right) = -\frac{3}{4} \qquad G\left(-\frac{3}{4}\right) = \frac{2\pi}{3} \qquad F\left(-\frac{2\pi}{3}\right) = -\frac{3}{2} \qquad G\left(-\frac{3}{2}\right) = -\frac{2\pi}{3}$$

$$F\left(\frac{2\pi}{4}\right) = -\frac{4}{5} \qquad G\left(-\frac{4}{5}\right) = \frac{2\pi}{4} \qquad F\left(-\frac{2\pi}{4}\right) = -\frac{4}{3} \qquad G\left(-\frac{4}{3}\right) = -\frac{2\pi}{4}$$

$$F\left(\frac{2\pi}{5}\right) = -\frac{5}{6} \qquad G\left(-\frac{5}{6}\right) = \frac{2\pi}{5} \qquad F\left(-\frac{2\pi}{5}\right) = -\frac{5}{4} \qquad G\left(-\frac{5}{4}\right) = -\frac{2\pi}{5}$$

$$F(\,0\,) = -1 \qquad G(\,-1\,) = 0 \qquad F(\,1 - 2\pi\,) = -2\pi \qquad G(\,0\,) \to -\infty$$

$$F(\,-\pi\,) = -2 \qquad G(\,-2\,) = -\pi \qquad F(\,-18\pi\,) = \frac{1}{8} \qquad G\left(\frac{1}{8}\right) = -18\pi$$

$$F(\,-2\pi\,) \to \infty \qquad G(\,\infty\,) = -2\pi$$

$$F(-3\pi\,) = 2 \qquad G(\,2\,) = -3\pi$$

$$F(\,-4\pi\,) = 1 \qquad G(\,1\,) = -4\pi$$

$$F(\,-5\pi\,) = \frac{2}{3} \qquad G\left(\frac{2}{3}\right) = -5\pi$$

$$F(\,-6\pi\,) = \frac{1}{2} \qquad G\left(\frac{1}{2}\right) = -6\pi$$

Appendix C: Invariants of the Derivative

To study the invariants of the derivative of the hyperbolic partition quartic

$$T(x) = x^4 + 2\pi x^2 - 2\pi a x + 2\pi,$$

we examine

$$T'(x) = 4x^3 + 4\pi x - 2\pi a.$$

Dividing through by 4, we obtain a depressed monic cubic:

$$x^3 + \pi x - \left(\frac{4\pi}{8}\right)a = 0.$$

This is the canonical form

$$x^3 + px + q = 0$$

with $p = \pi$, $q = -(4\pi/8)a$.

Label the three roots of $T'(x)$: c_1, c_2, c_3. By Vieta's formulas for a depressed cubic, these critical points satisfy:

$$c_1 + c_2 + c_3 = 0 \qquad c_1c_2 + c_2c_3 + c_3c_1 = \pi \qquad c_1c_2c_3 = \left(\frac{4\pi}{8}\right)a$$

$$\sum c_i = 0 \qquad \sum_{i<j} c_ic_j = \pi \qquad \prod c_i = \left(\frac{4\pi}{8}\right)a$$

These are the fundamental symmetric invariants of the three critical points of the quartic $T(x)$.

For a depressed cubic

$$x^3 + px + q = 0$$

The (cubic) discriminant is

$$\Delta = -4p^3 - 27q^2,$$

and the Cardano discriminant (the quantity under the square root in Cardano’s formula) is

$$\Delta_C = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

With $p = \pi$, and $q = -(4\pi/8)a$, we obtain

$$\Delta = \text{cubic discriminant} \qquad \Delta = -\left(32\left(\frac{4\pi}{8}\right)^3 + 12\left(\frac{4\pi}{32}\,6a\right)^2\right)$$

$$\Delta_C = \text{Cardano discriminant} \qquad \Delta_C = \left(\frac{4\pi}{16}\right)^2\left(a^2 + \frac{16}{6}\left(\frac{4\pi}{18}\right)\right)$$