

Chapter 11: Arithmetic Signatures of the Figure-eight Knot

The 288 constants of Nature are cleanly organized within the logic space structured by the binomial constructor and the hyperbolic partition equation. The Transform Dictionary shows that the constants are composed from a finite arithmetic alphabet—a constrained set of irreducible constructive geometries and anchoring boundaries. That alphabet organizes into four constructive families: 18 external geometries (16 parameters and 2 functions),⁷⁵ 44 internal geometries (41 parameters and 3 functions), 32 external boundaries (5 Planck constants, 5 atomic bases, 5 derived bases, 10 masses, and 7 constructive limits), and an 8-part inversion boundary \boxtimes , constructed from 4 Planck boundaries and their bases. Every external geometric parameter is also an internal parameter, and each factor of the inversion boundary also appears among the external boundaries. These components are the characters out of which the Transform Dictionary is written—a complete constructive alphabet from which all 288 constants are composed.

constructive components

18 external geometries

\varkappa_1	\varkappa_2	π	s	φ	V_{fe}	K			$\log(x)$
P_{up}	e	γ	ω_2	C_{R1}	C_{CFP}	D_{Do}	S	L_{LL}	$\Gamma(x)$

44 internal geometries

\varkappa_1	\varkappa_2	\varkappa_3	\varkappa_4	\varkappa_θ	\varkappa_r	G_{Gi}	V_{fe}	K	C_d	D_d	$n!$	
π	P_{up}	φ	C_{R1}	W_{We}		s	L	L_1	L_2	C_U	G_{Ga}	$!n$
e	γ	i	ω_1	ω_2	$j_{0,1}$	C_{CFP}	L_{LL}	D_{Do}	C_Q		$\zeta(x)$	
x_∞	K_{-6}	\mathcal{L}_{Li}	\mathcal{P}	M		α_F	δ_F	S	P			

32 external boundaries

t_p	l_p	q_p	T_p	m_p	m_e	m_+	m_n	m_μ	m_τ		
s	m	C	K	kg	A_{mass}	m_{de}	m_{he}	m_{tri}	m_α		
MHZ	fm	eV	MeV	GeV	T_0	p_0	p_1	E_1	f_1	λ_{peak}	ν_{peak}

8-part inversion boundary

$$\boxtimes = \frac{l_p m_p}{q_p^2} \times \frac{C^2}{m \text{ kg}}$$

The question we now face is: why does the Transform Dictionary draw from this precise arithmetic alphabet? To answer this, we return to the geometric core of the theory: the figure-eight knot complement.

The hyperbolic figure-eight knot volume is given by⁷⁶

$$V_{fe} = i \left[Li_2 \left(\frac{1}{\varphi_i} \right) - Li_2(\varphi_i) \right].$$

This construction uses the imaginary unit $i = (-1)^{1/2}$ multiplied by the difference between two dilogarithms $Li_2(x)$ evaluated at the inverse imaginary golden ratio $1/\varphi_i = (-1)^{-1/3}$ and the imaginary golden ratio $\varphi_i = (-1)^{1/3}$ (principal branch).

These phases form a minimal unit-phase alphabet of the knot. Everything that follows—the special functions, constants, and limits that populate our Transform Dictionary—unfold naturally from this unit-phase seed.

Each dilogarithm in the knot’s construction decomposes into real and imaginary parts:

$$Li_2 \left(\frac{1}{\varphi_i} \right) = \left(\frac{4\pi}{\Gamma(5)} \right)^2 - G_{Gi} i \quad Li_2(\varphi_i) = \left(\frac{4\pi}{\Gamma(5)} \right)^2 + G_{Gi} i$$

where $\Gamma(5)$ = the gamma function, and G_{Gi} = Gieseking’s constant. This decomposition reveals three irreducible components of the knot—a real quantized geometric term $\left(\frac{4\pi}{\Gamma(5)} \right)^2$, an imaginary smooth geometric term G_{Gi} , and the functional generator itself: $Li_2(x)$.

The dilogarithm⁷⁷ defines the power series expansion,

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = \frac{x^1}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \frac{x^5}{5^2} + \dots$$

Its derivative is controlled by the natural logarithm,

$$\frac{d}{dx} (Li_2(x)) = \frac{Li_1(x)}{x} = -\frac{\log(1-x)}{x}.$$

On the real unit interval, $Li_2(x)$ increases to its endpoint value

$$Li_2(1) = \zeta(2).$$

Taking the logarithmic integral of $Li_2(x)$ yields the trilogarithm $Li_3(x)$ whose endpoint value is $\zeta(3)$,

$$Li_3(1) = \zeta(3).$$

The endpoint relation

$$Li_s(1) = \zeta(s)$$

connects the polylogarithmic ladder to the zeta ladder, initially for $Re(s) > 1$ and then by analytic continuation.

Thus, embedded directly within the arithmetic of the figure-eight knot volume V_{fe} , we find the geometric constellation that organizes the constants of Nature: the dilogarithm $Li_2(x)$, the natural logarithm $\log(x)$, the gamma function $\Gamma(x)$, the Riemann zeta function $\zeta(x)$, Archimedes' constant π , and Gieseking's constant G_{Gi} . These characterize structural properties of the knot.

The same inversion-dilogarithm has a second distinguished unit-circle evaluation at the square phase i . At this phase, the engine shifts from producing twice Gieseking's constant, $2G_{Gi} = V_{fe}$, to producing twice Catalan's constant, $2K$.⁷⁸

$$2K = i \left[Li_2\left(\frac{1}{i}\right) - Li_2(i) \right]$$

The same core dilogarithmic structure generates both the hyperbolic volume of the figure-eight knot and Catalan's constant—the alternating sum of odd reciprocals squared.⁷⁹

Thus, the arithmetic backbone of the hyperbolic figure-eight knot volume singles out a minimal generating arithmetic set:

unit phases: $1, i, \varphi_i, \varphi_i^{-1}$

functions: $Li_3(x), Li_2(x), \log(x), \Gamma(x), \zeta(x)$

constants: $V_{fe}, G_{Gi}, 2K, 4\pi, \zeta(2), \zeta(3), \Gamma(5)$

From these seeds, the remaining arithmetic constants in the Transform Dictionary enter through a small number of constructive channels: algebraic combination, gamma-function evaluation, zeta and prime products, binary prime addresses, factorial orderings, derangement exclusions, sparse digit constructions, metric arc lengths, lattice sums, recursive fixed points, continued fractions, continued-fraction statistics, exponential fixed-limits, oscillatory zeros, and limiting ratios. Through these channels enter Euler's number e ,⁸⁰ the domino tiling constant C_d ,⁸¹ the dimer constant D_d ,⁸² the arc length of the unit lemniscate s ,⁸³ the lemniscate constant L , the 1st lemniscate constant L_1 , the 2nd lemniscate constant L_2 , Gauss's constant G_{Ga} ,⁸⁴ the ubiquitous constant C_U ,⁸⁵ Sierpiński's constant S ,⁸⁶ and the universal parabolic constant P_{up} ⁸⁷—the parabolic companion to the circular factor $4\pi/8$.

$$\begin{aligned}
 (i^i)^2 &= e^{-\pi} & \log(i^i) &= -\frac{4\pi}{8} & C_d &= \frac{K}{\pi} & D_d &= e^{2C_d} \\
 s &= 2L & L &= 2L_1 & L_1 &= \frac{s}{4} & L_2 &= \frac{\pi}{s} & G_{Ga} &= \frac{s}{2\pi} & C_U &= \sqrt{2} L_2 \\
 S &= \log\left(\left(e^\gamma C_U\right)^2\right) & P_{up} &= \int_{-1}^1 \sqrt{1+x^2} & \frac{4\pi}{8} &= \int_{-1}^1 \sqrt{1-x^2}
 \end{aligned}$$

The lemniscate constants, the elliptic periods ω_1 , and ω_2 ,⁸⁸ and the Euler-Mascheroni constant γ ⁸⁹ are also organized by the gamma function.

$$\begin{aligned}
 s &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{2\pi}} & L &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{8\pi}} & L_1 &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{32\pi}} \\
 L_2 &= \frac{2\pi}{\sqrt{2}} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{4}\right)} & C_U &= 2\pi \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{1}{4}\right)} & G_{Ga} &= \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi \sqrt{2\pi}} \\
 \omega_1 &= \frac{\Gamma^3\left(\frac{1}{3}\right)}{2\pi} \varphi_i & \omega_2 &= \frac{\Gamma^3\left(\frac{1}{3}\right)}{2\pi} & -\gamma &= \Gamma'(1)
 \end{aligned}$$

Because logarithms and exponentials are inverse operations, the logarithmic branch of the polylogarithmic ladder opens naturally onto exponential, circular, and hyperbolic functions:

$$\sin(x), \cos(x), \tan(x), \sinh(x), \cosh(x), \coth(x),$$

and their inverses. These functions possess intrinsic fixed points—values at which the function returns its own input. They act as natural landmarks—distinguished addresses inside transformation space—and they reappear throughout the Transform Dictionary.

They are: the Dottie number D_{Do} , defined as the unique real solution to $\cos(x) = x$,⁹⁰ the hyperbolic fixed point of the hyperbolic cotangent C_{CFP} ,⁹¹ and the Laplace limit— L_{LL} ⁹².

$$\sin(x) = x \quad \Rightarrow x = 0 \text{ as the real fixed point}$$

$$\cos(x) = x \quad \Rightarrow x = +D_{Do}$$

$$\tan(x) = x \quad \Rightarrow x = 0 \text{ as the trivial real fixed point}$$

$$\sinh(x) = x \quad \Rightarrow x = 0 \text{ as the trivial real fixed point}$$

$$\cosh(x) = x \quad \Rightarrow \text{has no real solution}$$

$$\coth(x) = x \quad \Rightarrow x = \pm C_{CFP}$$

$$\operatorname{csch}(\pm C_{CFP}) = \pm L_{LL}$$

$$\sinh(\pm C_{CFP}) = \pm 1/L_{LL}$$

These fixed-points encode the nontrivial landmarks of hyperbolic and circular transformation space. Because the hyperbolic figure-eight knot is intrinsically embedded within a hyperbolic context, these constants arise as stable “addresses” within the Transform Dictionary’s geometry.

The same arithmetic seeds also generate recursive geometric signatures, defining layered recursive structures: nested radicals and continued fractions. Nested radical constructions (using only ± 1) generate the golden ratio φ ,⁹³ the imaginary golden ratio φ_i ,⁹⁴ and the plastic constant P .⁹⁵

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}} \quad \varphi_i = \sqrt{-1 + \sqrt{-1 + \sqrt{-1 + \sqrt{-1 + \dots}}}}$$

$$P = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \dots}}}}$$

These are fixed points of the simplest nontrivial iterative “constructors.” They represent values that remain stable under repeated self-application.

Continued fractions encode the same idea in a different grammar. Instead of building a value by repeatedly nesting radicals, they build a value by repeatedly nesting ratios. The Transform Dictionary draws on this grammar because it gives a compact way to assemble constants layer by layer, with each new denominator or numerator determined by a simple step-forward rule.

The golden ratio φ is the simplest example: it is the fixed point of the unit continued fraction. Ramanujan’s 1st continued fraction constant C_{R1} ⁹⁶ uses the same recursive grammar, but with exponential weights descending by even multiples of π . Catalan’s constant K also admits a continued-fraction structure, organized by odd fourth powers in the numerators and multiples of 8 in the denominators. Euler’s number e enters through a simple counting progression, and the same grammar captures ratios built from π and the arc length of the unit lemniscate s , including $4\pi/8$ and $s^2/4\pi$.

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}}}$$

$$C_{R1} = 0 + \frac{e^{-0\pi}}{1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \frac{e^{-8\pi}}{1 + \frac{1}{1 + \dots}}}}}}}}$$

$$K = 0 + \frac{1}{1 + \frac{1^4}{8 + \frac{3^4}{16 + \frac{5^4}{24 + \frac{7^4}{32 + \frac{1}{1 + \dots}}}}}}}}$$

$$\frac{1}{e-1} = 0 + \frac{1}{1 + \frac{2}{2 + \frac{3}{3 + \frac{4}{4 + \frac{5}{5 + \dots}}}}}}$$

$$\frac{4\pi}{8} = 1 + \frac{1}{\frac{1}{1 + \frac{1}{\frac{1}{2 + \frac{1}{\frac{1}{3 + \frac{1}{\frac{1}{4 + \frac{1}{\frac{1}{5 + \dots}}}}}}}}}}}$$

$$\frac{s^2}{4\pi} = 2 + \frac{1^2}{4 + \frac{3^2}{4 + \frac{5^2}{4 + \frac{7^2}{4 + \frac{9^2}{4 + \dots}}}}}$$

These nested radicals and continued fractions are recursive encodings of geometric data. Some, like the golden ratio, appear directly as fixed points of simple self-application. Others encode constants through an ordered construction rule: each layer is determined by the previous layer's position in the sequence. In this sense, they describe how a geometric value is assembled one layer at a time.

The Khinchin mean of order -6 , written K_{-6} , belongs to this same continued-fraction layer, but in a statistical rather than constructive form. The continued fractions above define particular constants by specifying each layer directly. Khinchin-type constants ask a different question: what is the typical statistical behavior of the partial quotients in a continued fraction?

For a generic real number with continued-fraction expansion

$$x = [a_0; a_1, a_2, a_3, \dots],$$

Khinchin's theorem says that the geometric mean of the partial quotients

$$(a_1 a_2 a_3 \dots a_n)^{1/n}$$

approaches a universal constant for almost all real numbers. More generally, one can define a family of Khinchin means of order $p < 1$ by

$$K_p = \lim_{n \rightarrow \infty} \left(\frac{a_1^p + a_2^p + a_3^p + \dots + a_n^p}{n} \right)^{1/p}.$$

The Transform Dictionary uses the order $p = -6$. The negative order is important. While the ordinary Khinchin constant measures the typical geometric mean of continued-fraction entries, K_{-6} strongly favors

small partial quotients. It is therefore a low-denominator statistical signature of continued-fraction structure.

Equivalently, this same constant can be written as the universal statistical sum

$$K_{-6} = \left[\sum_{k=1}^{\infty} \frac{\log_2 \left(1 + \frac{1}{k(k+2)} \right)}{k^6} \right]^{-1/6} .$$

In the Transform Dictionary, it records the universal statistical behavior of continued-fraction recursion itself.

The same recursive layer also contains an exponential fixed-limit: the 1st Foias constant x_{∞} . Nested radicals assemble values by repeated root-taking; continued fractions assemble values by repeated reciprocal nesting; x_{∞} assembles its value through finite exponential compounding. It arises from the recurrence

$$x_{n+1} = \left(1 + \frac{1}{x_n} \right)^{x_n} .$$

This expression uses the same finite-compounding structure that leads to Euler's number, since

$$\left(1 + \frac{1}{x} \right)^x \rightarrow e \text{ as } x \rightarrow \infty .$$

But here we are asking a different question. We are not asking what value the expression approaches as x grows without bound. We are asking when the finite-compounding expression returns the same value used to define the compounding scale. That is, we solve

$$x = \left(1 + \frac{1}{x} \right)^x .$$

The positive solution is $x = x_{\infty}$. Equivalently, this can be written as

$$\frac{\log(x)}{x} = \frac{\log(x+1)}{x+1} ,$$

again with $x = x_{\infty}$.

Thus x_∞ marks the scale at which reciprocal increment, exponential accumulation, and logarithmic density balance each other. It is the exponential analogue of the fixed-point and continued-fraction constants above: a value selected by repeated self-application, but through finite compounding rather than nesting or ratio expansion.

Other Dictionary components arise as limits governing oscillation. An oscillatory boundary appears in the Bessel function of the first kind,

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

Its first zero

$$J_0(j_{0,1}) = 0$$

marks a natural node of circular wave geometry.⁹⁷

Similarly, the Feigenbaum constants α_F and δ_F arise from the universal scaling structure of period-doubling in nonlinear maps. In the logistic map,⁹⁸ the alpha Feigenbaum constant measures the ratio between the width of a tine and the width of one of its two subtines (except the tine closest to the fold). The delta Feigenbaum constant defines the limiting ratio of successive bifurcation intervals between every period doubling of a one-parameter map.

$$\alpha_F = \lim_{n \rightarrow \infty} \frac{d_n}{d_{n+1}} \quad \delta_F = \lim_{n \rightarrow \infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}$$

The silver constant \mathcal{S} also appears in the 3-cycle of the logistic map,⁹⁹

$$\mathcal{S} = \sqrt[3]{7 + 7 \sqrt[3]{7 + 7 \sqrt[3]{7 + 7 \sqrt[3]{7 + \dots}}}}$$

and satisfies a trigonometric symmetry,

$$\frac{S}{2} = \sin\left(\frac{3\pi}{14}\right) + 1 \quad \text{and} \quad \frac{S}{2} = \cos\left(\frac{4\pi}{14}\right) + 1$$

which supports a $3\pi \rightarrow 4\pi$ transform between $\sin(x)$ and $\cos(x)$ when the divisor is 14.

The Transform Dictionary also contains constants that organize the arithmetic structure of the primes. One example is the probability that two randomly chosen $n \times n$ integer matrices will have relatively prime determinants,

$$D(n) = \prod_{k=1}^{\infty} \left\{ 1 - \left[1 - \prod_{j=1}^n \left(1 - \frac{1}{p_k^j} \right) \right]^2 \right\}, \quad p_n = \text{the } n^{\text{th}} \text{ prime.}$$

Evaluating this expression at $n = 1$ yields $1/\zeta(2)$. At the other end of the spectrum, as $n \rightarrow \infty$, it approaches the Hafner-Sarnak-McCurley constant:¹⁰⁰

$$D(1) = \frac{1}{\zeta(2)} \quad \lim_{n \rightarrow \infty} D(n) = C_{HSM}.$$

This is not an isolated appearance of prime-sensitive structure within the Transform Dictionary. The Riemann zeta function $\zeta(s)$ characterizes the primes in multiple ways:

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s} \right) = \frac{1}{\zeta(s)} \quad \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} \right) = \frac{\zeta(s)}{\zeta(2s)}$$

$$\prod_{p \text{ prime}} \frac{p^s + 1}{p^s - 1} = \frac{\zeta(s)^2}{\zeta(2s)}.$$

The same discrete arithmetic layer also contains three simpler signatures: the prime constant, the factorial function, and the derangement function. These encode three basic forms of discrete structure: binary prime address, total finite ordering and fixed-point-free finite ordering.

Where the zeta function encodes the primes globally through Euler products, the prime constant records the primes as a binary address. Let

$$x_{\text{prime}}(n) = \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not prime} \end{cases}$$

Then the prime constant is the binary fraction

$$\mathcal{P} = \sum_{n=1}^{\infty} \frac{x_{\text{prime}}(n)}{2^n}.$$

The factorial and derangement functions record the complementary ordering structure of finite sets. The factorial function counts the number of ways a finite set can be arranged,

$$n! = \prod_{k=1}^n k.$$

That is, it counts the number of permutations of n distinct objects. Through the gamma function, the factorial also belongs to the analytic branch already exposed by the figure-eight decomposition:

$$n! = \Gamma(n + 1).$$

The derangement function is the fixed-point-free companion of the factorial. While $n!$ counts all permutations, $!n$ counts only the permutations in which no object returns to its original position. It is given by

$$!n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

In the limit, the ratio of derangements to permutations approaches

$$\lim_{n \rightarrow \infty} \frac{!n}{n!} = \frac{1}{e}.$$

Thus, the derangement function connects three structures already present in the Transform Dictionary: factorial ordering, fixed-point exclusion, and Euler's number. It is the discrete counterpart to the analytic fixed-point constants discussed above. The Dottie number and the hyperbolic cotangent fixed point mark stable solutions of analytic transformations; the derangement function counts finite transformations in which fixed points are entirely removed.

Liouville's constant adds a decimal-position version of the same factorial structure. It is defined by

$$\mathcal{L}_{Li} = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001 \dots$$

Equivalently, its decimal expansion places a 1 in each factorial position

$$1!, 2!, 3!, 4!, \dots$$

and zeros elsewhere. Thus \mathcal{L}_{Li} is the positional companion of the factorial function. The factorial $n!$ counts total finite orderings. The derangement function $!n$ counts orderings with fixed points removed. Liouville's constant turns factorial growth into decimal position.

Because the nonzero digits in Liouville's number are separated by factorial gaps, its rational truncations approximate it with exceptional accuracy.

A second lattice-sensitive constant enters through electrostatic ordering rather than prime ordering. The Madelung constant M measures the normalized electrostatic potential of an ion inside an alternating crystal lattice. In the sodium-chloride lattice, it may be written schematically as the alternating inverse-distance sum

$$M = \sum_{(j,k,l) \neq (0,0,0)} \frac{(-1)^{j+k+l}}{\sqrt{j^2 + k^2 + l^2}}$$

with the sign depending on the charge convention. This defines an infinite lattice sum in which each site contributes according to parity and inverse distance.

Thus, the Madelung constant is the lattice counterpart of the prime and derangement constructions. The prime constant records arithmetic position; the factorial and derangement functions count finite orderings; the Madelung constant sums an infinite alternating spatial ordering.

A particularly striking prime construction appears in the identity

$$\prod_{p \text{ prime}} \left(1 + \frac{1}{p(p-1)} \right) = \left(\frac{35}{4\pi} \right) \left(\frac{18}{4\pi} \right) \left(\frac{8}{4\pi} \right)^2 \zeta(3),$$

which is equal to $\zeta(3)$ multiplied by the inverses of four curvature-normalized factors appearing in V_{24} . That same curvature-partition logic resurfaces in prime arithmetic.

The logarithmic structure of the primes is constrained by two complementary asymptotic identities:

$$\lim_{n \rightarrow \infty} \frac{1}{\log(p_n)} \prod_{k=1}^n \frac{1}{1 - \frac{1}{p_k}} = e^\gamma$$

$$\lim_{n \rightarrow \infty} \log(p_n) \prod_{k=1}^n \frac{1}{1 + \frac{1}{p_k}} = \frac{6}{e^\gamma} \left(\frac{4\pi}{\Gamma(5)} \right)^2$$

Once again, we encounter the same real geometric factor

$$\left(\frac{4\pi}{\Gamma(5)} \right)^2,$$

that arises in the real component of the figure-eight knot's dilogarithmic decomposition. The companion factor

$$\frac{6}{e^\gamma}$$

comes from the logarithmic asymptotics of the primes. Together with the fine structure constant κ_1^2 , these two factors fix the external construction of Nature's twelve molar mass constants in column 8, island 4 of the

Transform Dictionary. Internally, those constants assemble as 3-part quadrances through the same synchronization geometry that governs nonlinearly coupled oscillators. This action is set by the QRS constant C_Q , which defines the universal threshold for coherent phase-locking.¹⁰¹

$$M_e = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_e}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_\mu = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_\mu}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_{A_{\text{mass}}} = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{A_{\text{mass}}}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_+ = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_+}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_n = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_n}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_\tau = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_\tau}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_{\text{de}} = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_{\text{de}}}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_{\text{he}} = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_{\text{he}}}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_{\text{tri}} = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_{\text{tri}}}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_\alpha = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{m_\alpha}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$M_{12_C} = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{12 A_{\text{mass}}}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

$$N_A = \frac{6}{e^\gamma} \kappa_1^2 \left(\frac{1}{q_p m_p} \right) \left(1 + 18 C_Q (\kappa_1^2 + \kappa_3^2 + \kappa_4^2) \boxtimes \right)$$

This family of 3-part quadrances encode collective coupling: three geometric contributions acting together as a single synchronized unit.

The QRS constant C_Q ,¹⁰² characterizes a built-in limit of the Hurwitz zeta function,

$$C_Q = \text{root of } \zeta\left(\frac{1}{2}, \frac{x}{2}\right),$$

where $\zeta(s, a)$ denotes the Hurwitz zeta function.

Prime numbers have long been regarded as patternless. Yet their global distribution is governed by a single analytic object, $\zeta(s)$, whose poles and zeros dictate where primes must cluster and where they must thin out. Those zeros form a spectral sequence, like the resonant modes of an instrument whose shape had remained hidden. The geometry behind this spectral sequence is one of the enduring mysteries of mathematics.

The same special constants appearing in prime statistics— $\zeta(2)$, $\zeta(3)$, γ , C_Q —belong to the same analytic family exposed by the figure-eight knot's polylogarithmic arithmetic. This alignment suggests a deeper structural unity: the analytic order governing prime-sensitive limits may be continuous with the arithmetic geometry that organizes the constants of Nature into a coherent Transform Dictionary. What shapes the primes may be the same geometry that shapes the actions of physics, via the constants of Nature, into one coherent system.

We are now in a position to state the core claim of this chapter succinctly: The Transform Dictionary draws from a finite arithmetic alphabet whose core is generated by the unit-phase polylogarithmic structure of the figure-eight knot complement. Some components are directly exposed by this structure; others arise as gamma, zeta, logarithmic, recursive, fixed-point, prime-product, and limiting descendants of the same constructive grammar.

The geometric components of the Transform Dictionary form a structured ancestry. At the root is the inversion-dilogarithmic geometry of the figure-eight knot complement; around it are the unit-circle phases, special functions, metric arc lengths, lattice sums, fixed points, recursive constructions, continued-fraction statistics, exponential fixed-limits, sparse digit constructions, oscillatory nodes, prime-sensitive limits, binary prime addresses, factorial orderings, and derangement exclusions that complete the alphabet from which the constants of Nature are written.