Chapter 12: Matrices—the Grammar of Transformation

If geometry encodes the nouns of reality—the structures that persist—then matrices encode its verbs: the lawful transformations by which one structure becomes another without losing coherence. They provide the natural language for transformation—not merely as motion through space, but as an algebra of relations governing how the parts of a system act together.

In this chapter we examine the hyperbolic partition equation as an operator structure. When the quartic is translated into matrix form, its roots become eigenvalues, its Vieta relations become matrix invariants, and its directed asymmetry becomes a generator of first-order evolution. The partition structure is no longer only a relation among roots; it becomes an operator structure whose spectrum, traces, determinants, eigenblocks, and generators organize the system's dynamics.

In the 19th century, Cayley and Sylvester recognized that transformations themselves could be treated as objects—things that could be added, multiplied, or inverted. This insight detached geometry from any particular coordinate system and established matrices as a universal grammar of transformation and change.[103]

A matrix mediates change. It encodes the linear couplings between components, revealing how each part of one structure contributes to the next. Through this language, matrices express coherent structural relations algebraically.

The simplest invariant of a matrix is its determinant, which measures how a transformation scales volume. If $\det = 2$, then the volume doubles. If $\det = 0$, the entire space collapses onto a lower-dimensional subspace.[104] Determinants therefore quantify how volume is preserved, expanded, or collapsed under transformation.

A real matrix with $\det = 1$ preserves volume. An integer matrix with $\det = 1$ does something stronger: it preserves the integer lattice itself. Such matrices are called unimodular. They reshuffle a lattice without changing the volume of its fundamental cell.[105]

A lattice may therefore be understood as space discretized by a matrix: the determinant gives the volume of the fundamental cell—the repeating tile from which the entire discrete structure is built. Every invertible matrix defines a tiling of space, and unimodular matrices are the volume-preserving reshufflers of such tilings.

A square matrix is invertible precisely when its $\det \neq 0$. Its eigenvalues—the roots of its characteristic polynomial—reveal the directions that are stretched, compressed, rotated, or phased by the transformation. The trace sums the eigenvalues, and for infinitesimal generators it measures the local tendency of a flow to expand or contract.

Among all matrices, the Hermitian ones play a special role: their eigenvalues are real, and this makes them suitable for representing observable quantities.[106] They keep the algebra of transformation tethered to *measurement*. When multiplied by i and exponentiated, Hermitian matrices generate unitary ones—transformations that preserve inner products and probability amplitudes, just as unimodular matrices preserve lattice volume.

In this sense, unimodular and unitary transformations play analogous roles in different geometries: one preserves the volume structure of a lattice, while the other preserves the inner-product structure of quantum states. Both express coherence through invariant-preserving transformation.

Before matrices were conceived, Lagrange recognized that physical systems evolve along paths selected by stationary *action*,[107]

$$S = \int L\, dt.$$

Hamilton reformulated this principle as a system of coupled flows in phase space—linking energy to time, and momentum to position.[108] His equations describe transformations that preserve the symplectic structure of phase space, a geometric precursor to the operator transformations that later preserve probability in quantum theory.

The Lagrangian encodes the action principle that selects physical paths; the Hamiltonian generates its coherent time evolution. Together they describe the self-consistent flow of transformation—a precursor of the operator equations that govern quantum mechanics.

> "Time is said to flow on, but the law of its flow is expressed by algebraic symbols."
> —William Hamilton[109]

In classical mechanics, these flows are governed by Poisson brackets; in quantum mechanics, they become commutators, connected by Dirac's correspondence principle:[110]

$$\{A, B\} \longrightarrow \frac{1}{i\hbar}[\hat{A}, \hat{B}].$$

With this, transformation becomes an operator calculus—a universal grammar for the dynamics of physical systems.

In 1925, Heisenberg reimagined mechanics without trajectories. Instead of tracking particles through space, he tracked the algebra governing measurable transitions between states. Those transitions naturally assembled into arrays—matrices—whose multiplication encoded the interference of possibilities.

Born and Jordan formalized this insight into *matrix mechanics*,[111] where position and momentum no longer commute but obey

$$[X, P] = i\hbar\, I.$$

Every continuous symmetry has an infinitesimal generator—an element of a Lie algebra whose exponential traces out the full motion of that symmetry,[112]

$$e^{At}.$$

Even a simple generator can unfold into a rich global structure. A simple but illuminating example is the shift operator whose exponential produces Pascal's triangle as a lower-triangular matrix.[113]

$$e\wedge\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix}$$

This illustrates a general Lie-theoretic idea: infinitesimal generators, when exponentiated, produce global transformations. Rotations, boosts, and gauge symmetries all arise from this algebraic fabric. At the smallest scale, the Pauli matrices embody this structure in its simplest form: three 2×2 matrices that generate the spin-$1/2$ representation of rotations.

Schrödinger recognized that the Hamiltonian could act as a differential operator on a continuous field—the *wavefunction*:

$$\hat{H}\psi = E\psi.$$

This revealed matrix mechanics and wave descriptions as two conjugate projections of a single structure—one discrete in its algebra of transitions, the other continuous in its wave representation. The wavefunction is geometry written in probability; the matrix is probability written in algebra. Both describe persistence through transformation.

Schrödinger's wave equation united energy and probability, but treated time and space asymmetrically, violating the symmetry demanded by relativity. The Klein-Gordon equation restored relativistic symmetry, but its second-order time structure produced an indefinite probability density in the single-particle interpretation.

Dirac's insight was to seek a first-order equation in both space and time—fully compatible with relativity—by taking what is effectively the algebraic square root of spacetime geometry itself. To do this, he needed mathematical objects whose products reproduce spacetime's metric.[114]

The objects capable of doing this were matrices.

Dirac introduced four gamma matrices that encode the orthogonality of spacetime directions algebraically. Each gamma matrix acts like a geometric axis, bound by anticommutation relations that preserve the metric structure:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\, I.$$

This allowed Dirac to write down the simplest statement that lets a wave persist in spacetime without breaking the symmetries of relativity. With the constants restored, the Dirac equation is

$$\left(i\hbar c\, \gamma^\mu\, \partial_\mu - m_e c^2 \right)\psi = 0$$

where c sets the spacetime conversion scale, $\hbar$ sets the quantum action scale, and m_e sets the electron's rest-energy scale.

The spinor ψ is the minimal object that can live in this double-covered algebraic world. It transforms not like an ordinary vector, but through a double-valued representation: it requires two full rotations to return to its original state.

Matrices articulate the grammar of coherent transformation—encoding the linear couplings between inputs and outputs, and underwriting the syntax by which the Universe preserves structure through change.

We now apply this grammar to a concrete object.

The Companion Matrix

With this grammar of matrices in hand, we now return to the quartic system that governs our hyperbolic partitions,

$$T(x) = x^4 + 2\pi x^2 - 2\pi a x + 2\pi = 0$$

whose associated Frobenius companion matrix is[115]

$$M = \begin{pmatrix} 0 & 0 & 0 & -2\pi \\ 1 & 0 & 0 & a\,2\pi \\ 0 & 1 & 0 & -2\pi \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The fundamental invariants of M are its eigenvalues and the symmetric relationships they encode through traces and determinants. The eigenvalues of M are the four roots $\{ж_1, ж_2, ж_3, ж_4\}$ of $T(x)$.

The trace of M is the sum of its eigenvalues, $\operatorname{tr}(M) = 0$, while the traces of higher powers encode higher power sums; for example, $\operatorname{tr}(M^2) = -4\pi$. These traces provide coordinate-free access to the quartic's internal invariants.

The determinant of M is the product of its eigenvalues, 2π, and more generally, $\det(M^k) = (2\pi)^k$ for all integers k. By contrast, $\det(e^M) = e^{\operatorname{tr}(M)} = 1$, so the exponential e^M is a volume-preserving transformation in 4-dimensions. This reflects only the vanishing trace of M in the exponential map, not the volume rescaling induced by the discrete companion map which rescales volume by a factor of 2π at each tick.

The spectral radius of M is the largest eigenvalue modulus, determined by the modulus of the complex pair, $|ж_3| = |ж_4| = ж_r$.

eigenvalues: $ж_1, ж_2, ж_3, ж_4$

$\mathrm{tr}(M) = 0$ $\quad$ $\det(M) = 2\pi$

$\mathrm{tr}(M^2) = -4\pi$ $\quad$ $\det(M^k) = (2\pi)^k$ for all integers k

$\rho(M) = ж_r$ $\quad$ $\det(e^M) = e^{\mathrm{tr}(M)} = 1$

Since M is invertible, it admits a complex matrix logarithm on a chosen spectral branch. Let $\Lambda = \log(M)$. Choose the conjugate-symmetric branch for which $\mathrm{tr}(\Lambda) = \log(2\pi)$. Then $e^{\Lambda} = M$, and the continuous interpolation of the discrete dynamics is $M^t = e^{t\Lambda}$. At integer times $t = k$, this returns the ordinary matrix powers M^k. Unlike e^M, the interpolating flow $e^{\Lambda t}$ does not preserve volume. Its determinant is

$$\det(e^{t\Lambda}) = e^{\mathrm{tr}(\Lambda)\,t} = (\det(M))^t = (2\pi)^t.$$

Thus, volume scales continuously by a factor of $(2\pi)^t$, matching the discrete jumps $(2\pi)^k$ at integer times. The algebraic structure is clear: the discrete step M rescales volume in quantized jumps, while its logarithmic generator induces a smooth dilation whose integer-time restriction reproduces those jumps. This gives the partition system a useful operator interpretation: continuous geometry with discrete actions steps.

The coefficient $2\pi a \neq 0$ introduces an odd-power asymmetry into the recurrence. It breaks the even symmetry of the untwisted quartic and gives the recurrence a directed bias. Because the coefficients of $T(x)$ are real, any nonreal eigenvalues occur in complex-conjugate pairs. In the parameter regime relevant to our hyperbolic partitions, two eigenvalues form a complex-conjugate pair; together they span a real two-dimensional oscillatory plane, characterized by simultaneous rotation and radial dilation. The other two eigenvalues are real and generate a hyperbolic invariant subspace.

Therefore, M real-similar to a block form containing a two-dimensional oscillatory block and a two-dimensional hyperbolic block, yielding the invariant splitting

$$\mathbb{R}^4 \cong \mathbb{R}^2_{\mathrm{osc}} \oplus \mathbb{R}^2_{\mathrm{hyp}}.$$

Viewed this way, M acts as the discrete companion operator of the system, while $\Lambda = \log(M)$ functions as its continuous generator. Together, the quartic $T(x)$ and its companion matrix define a clockable operator architecture: a system whose geometry can be modeled continuously, while its action advances in discrete spectral steps.

Decomposing the Quartic

Ferrari's resolvent cubic[116] naturally pairs the four roots $\{ж_1, ж_2, ж_3, ж_4\}$ into two quadratic factors, each with real coefficients. The real root pair forms one quadratic factor, while the complex-conjugate pair forms the other. Each quadratic therefore admits its own 2×2 companion matrix.

$$M_+ = \begin{pmatrix} 0 & -ж_1ж_2 \\ 1 & ж_1 + ж_2 \end{pmatrix} \qquad M_- = \begin{pmatrix} 0 & -ж_3ж_4 \\ 1 & ж_3 + ж_4 \end{pmatrix}$$

Their traces and determinants encode the local Vieta data, preserving the algebraic invariants of each root pair.

$$\begin{aligned} \det(M_+) &= ж_1ж_2 & \det(M_-) &= ж_3ж_4 \\ \mathrm{tr}(M_+) &= ж_1 + ж_2 & \mathrm{tr}(M_-) &= ж_3 + ж_4 \\ \mathrm{tr}\left({M_+}^2\right) &= {ж_1}^2 + {ж_2}^2 & \mathrm{tr}\left({M_-}^2\right) &= {ж_3}^2 + {ж_4}^2 \end{aligned}$$

There are three possible ways to partition four roots into two disjoint pairs. Ferrari's method selects the unique pairing that yields a real quadratic factorization: the pairing that respects the real structure of the system.

The Oscillatory Block

When a quartic possesses a complex-conjugate pair of roots

$$ж_3, ж_4$$

the dynamics associated with that pair live inside a real two-dimensional oscillatory plane. Any nonreal eigenvalues of the companion matrix therefore generate a rotation-scaling block inside its real block decomposition.

Write the complex pair as

$$ж_3 = Re(ж_3) + Im(ж_3)\, i, \qquad ж_4 = Re(ж_4) + Im(ж_4)\, i,$$

with $Re(ж_3) = Re(ж_4)$, and $Im(ж_3) = -Im(ж_4)$.

To isolate this subsystem, we write the oscillatory block in two conjugate orientations:

$$M_{osc}{}^{+} = \begin{pmatrix} Re(ж_3) & -Im(ж_3) \\ Im(ж_3) & Re(ж_3) \end{pmatrix},$$

$$M_{osc}{}^{-} = \begin{pmatrix} Re(ж_4) & -Im(ж_4) \\ Im(ж_4) & Re(ж_4) \end{pmatrix}.$$

These two matrices describe the same oscillatory plane with opposite orientation. The $ж_3$-form rotates forward; the $ж_4$-form rotates backward. Both have the same eigenvalue pair, $ж_3$ and $ж_4$.

In polar form $ж_3 = ж_r\, e^{ж_\theta i}$ and $ж_4 = ж_r\, e^{-ж_\theta i}$. Therefore, the two orientation blocks become

$$M_{osc}{}^{+} = ж_r \begin{pmatrix} cos(ж_\theta) & -sin(ж_\theta) \\ sin(ж_\theta) & cos(ж_\theta) \end{pmatrix},$$

$$M_{osc}{}^{-} = ж_r \begin{pmatrix} cos(ж_\theta) & sin(ж_\theta) \\ -sin(ж_\theta) & cos(ж_\theta) \end{pmatrix}.$$

The $+$ block performs a rotation by $+ж_\theta$, together with a dilation by $ж_r$. The $-$ block performs the conjugate rotation by $-ж_\theta$, with the same dilation factor. Their fundamental invariants are identical:

$$\det(M_{osc}) = ж_3 ж_4 = ж_r{}^2,$$

$$\mathrm{tr}(M_{osc}) = ж_3 + ж_4 = 2Re(ж_3).$$

The phase advance is determined by the argument $ж_\theta$, while the dilation rate is controlled by the modulus $ж_r$. The quartic's complex-conjugate pair therefore contributes a rotational-dilational subsystem: a plane whose dynamics combine phase advance and radial scaling as inseparable aspects.

The quadratic companion block M_{-} and either oriented form $M_{osc}{}^{\pm}$ are real-similar descriptions of the same complex-conjugate subsystem. The full companion matrix M is therefore real-similar to a block diagonal form built from two quadratic blocks: one hyperbolic, arising from the real root pair, and one oscillatory, arising from the complex pair. Each block carries its own local computational grammar. Together they form the complete four-dimensional companion realization governing hyperbolic partitions.

The Lagrangian

To construct the Lagrangian of this system, we begin by writing the differential operator corresponding to $T(x)$:

$$L = D^4 + 2\pi D^2 - 2\pi a D + 2\pi, \qquad D = \frac{d}{dt}.$$

Under time-reversal $t \longmapsto -t$, the operator $D \longmapsto -D$, so the time-reversed (adjoint) operator is

$$L^* = D^4 + 2\pi D^2 + 2\pi a D + 2\pi.$$

The odd-derivative term is the source of the directional asymmetry. A single-field quadratic action for $L[x] = 0$ cannot be both real and time-reversal symmetric when $a \neq 0$, because the odd-derivative term changes sign under time reversal and therefore cannot be paired with itself in a real one-field action.

To restore symmetry at the level of the action, we introduce two real fields $x(t)$ and $y(t)$, representing the forward and backward flows of the system, and combine them into the doubled action

$$\mathcal{L}(x, y) = \frac{1}{2}\Big(y\, L\,[x] + x\, L^*[y] \Big).$$

Varying the action $S = \int \mathcal{L}(x, y)\, dt$ with respect to y returns the forward equation $L\,[x] = 0$, while varying with respect to x returns the backward, adjoint equation $L^*[y] = 0$. Although each equation alone breaks time-reversal symmetry, their coupled action is real and invariant under $t \longmapsto -t$.

In this way, the forward law and its adjoint partner become two halves of one real variational structure.

Combining the pair (x, y) into a complex field makes a natural phase symmetry visible:

$$\psi = \frac{x + iy}{\sqrt{2}}, \qquad \psi \longmapsto e^{\theta i}\psi.$$

When the doubled action is written in phase-invariant bilinear form, Noether's theorem supplies the corresponding conserved quantity.[117] This two-field Lagrangian is therefore a minimal doubled action for the quartic dynamics of $T(x)$ that preserves the forward/backward pairing. It binds forward and backward laws into one real action, exposes the complex structure of the system, and prepares the transition to $U(1)$ phase symmetry.

When replicated across multiple components, the same construction provides a natural route toward $U(n)$ symmetry, and, with additional determinant-one or traceless constraints, toward $SU(n)$).

From Lagrangian Flows to First-order Evolution

Every fourth-order differential equation can be rewritten as a coupled first-order system. Define the four-component state vector[118]

$$\boldsymbol{u}(t) = \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \\ \dddot{x} \end{pmatrix}.$$

The quartic equation $L[x] = 0$ becomes a linear flow

$$\frac{d}{dt}\boldsymbol{u}(t) = H\,\boldsymbol{u}(t)$$

with generator

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\pi & 2\pi a & -2\pi & 0 \end{pmatrix}.$$

The characteristic polynomial of H is

$$\det(zI - H) = z^4 + 2\pi z^2 - 2\pi a z + 2\pi.$$

Its eigenvalues are exactly the four roots of the quartic $T(z) = 0$. This means that H and the discrete companion matrix M share the same spectral data: both are matrix realizations of the same quartic structure. H is the transpose companion realization.

$$H = M^{\mathrm{T}}$$

The matrix M and H are transposed realizations of the same quartic structure. The matrix M is the Frobenius companion matrix whose characteristic polynomial is $T(x)$, while H is the first-order state-evolution matrix associated with the corresponding recurrence or differential equation. Acting on a four-component state vector, H advances the system by one step in the recurrence, or generates continuous first-order flow in the differential formulation.

Because transposition preserves the characteristic polynomial, M and H share the same eigenvalue spectrum. They therefore preserve the same quartic geometry: the same roots, the same spectral invariants, and the same operator structure expressed in two complementary forms.

It is useful to split H into an even, unperturbed part and a directional interaction part:

$$H = H_0 + aY,$$

with

$$H_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2\pi & 0 & -2\pi & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2\pi & 0 & 0 \end{pmatrix}.$$

The matrix H_0 corresponds to the even, time-reversal-paired operator obtained by setting $a = 0$, while the interaction matrix Y inserts the odd-derivative contribution into the first-order flow. The coupling aY controls how strongly the first-derivative component $\dot{x}$ feeds into the fourth-derivative closure equation.

In this sense, aY acts as a finite-dimensional mixing term: it mixes derivative components and deforms the operator spectrum away from the even $a = 0$ form.[119] For a fixed value of a, the quartic $T(z)$ fixes the spectrum, varying a changes the directed mixing and therefore changes the spectral structure.

Once the doubled fields are packaged into a complex field with a global $\mathrm{U}(1)$ phase, the generator H becomes part of a larger operator grammar. The quartic $T(z)$ fixes the spectrum, while the coupling a controls the directed mixing between forward and backward sectors.

This is the algebraic outline of Hamiltonian or Schrödinger-type evolution: a linear, first-order generator acts on a finite-dimensional state space, and its exponential gives the continuous evolution operator,

$$\boldsymbol{u}(t) = e^{tH}\, \boldsymbol{u}(0).$$

The pairing (x, y) defines an internal complex structure; the doubled action supplies a bilinear pairing; the phase notation prepares the transition to $\mathrm{U}(1)$-type symmetry; and the exponential e^{tH} acts linearly on the state vector.

This structure is the minimal algebraic core of the operator evolution: a coherent first-order flow, a doubled variational pairing, and a natural route toward phase symmetry.

Clifford Structure From the Quartic Operator

The spectral structure revealed above possesses the kind of algebraic ingredients needed to linearize the dynamics. Ferrari's resolvent shows that the quartic operator L factors into two quadratic operators:

$$L = (D^2 + pD + q)(D^2 - pD + r),$$

with real coefficients p, q, and r.

Each quadratic operator can be represented as a first-order matrix system. Dirac's strategy was to seek a first-order matrix operator whose product with its conjugate reproduces the quadratic form. A quadratic expression can be hidden inside a first-order operator only if the matrices multiplying the first-order directions obey a definite anticommutation law.

If those matrices are denoted Γ^μ, their defining relation is

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} I.$$

This is the Clifford relation. It is the rule that lets first-order matrix generators preserve a second-order quadratic form.

Thus, the quartic operator hides a pair of quadratic operators. Each quadratic operator admits a first-order representation, and the pair together share the same square-root grammar that Clifford algebras formalize.

This is the same algebraic sequence that appears in Dirac's construction. Dirac began with the second-order Klein-Gordon structure,

$$\square + m^2,$$

and introduced a first-order operator,

$$i\gamma^\mu \partial_\mu - m,$$

whose product with its conjugate recovers the second-order equation.[120] The gamma matrices obey

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I,$$

so the geometry of spacetime is carried by matrix anticommutation.

The quartic construction follows this operator grammar in finite-dimensional form. The fourth-order operator factors into two quadratic pieces. Each quadratic piece admits a first-order square-root representation. Those representations introduce Clifford-type anticommutation relations.[121] The doubled field (x, y) carries the forward and backward sectors together in one algebraic object, giving it a spinor-like role. The generator H acts as a Dirac-like first-order operator on the finite-dimensional state space.

The hyperbolic partition structure of $T(x)$ therefore carries the same algebraic architecture that Dirac used to introduce gamma matrices and spinors. From a simple quartic polynomial encoding hyperbolic curvature exchange, we are led to a finite-dimensional version of the operator grammar that governs relativistic quantum theory. This gives us a disciplined structure to test: geometry generates an operator algebra, with the constants of Nature interpreted as the measurable invariants of its allowed transformations.