

Chapter 7: Invariants of the hyperbolic partition equation

“What is invariant under transformation is what is real.”
—Hermann Weyl⁴⁸

To reveal the hierarchy of algebraic symmetries built into the hyperbolic partition equation, we begin by examining the equation’s zero-parameter core. This stripped-down case allows the internal symmetry of the partition rule to be studied before the measured Planck mass scale offset is introduced. The core equation is

$$A(x) = \frac{1}{x} + x + \frac{x^3}{2\pi} = 0.$$

Multiplying through by $2\pi x$ gives the even quartic

$$P(x) = x^4 + 2\pi x^2 + 2\pi = 0.$$

Because the quartic contains only even powers of x , the substitution $t = x^2$ collapses this quartic into a quadratic.

$$Q(t) = t^2 + 2\pi t + 2\pi = 0.$$

Thus, the partition equation has quartic structure in x , quadratic structure in $t = x^2$, and Möbius fixed-point structure in the roots of that quadratic. To catalog the invariants of $Q(t)$ and $P(x)$ we use the standard machinery governing quadratic fixed points and quartic root relations.

Euler’s fixed-point (Möbius) maps

In 1748, Euler observed that a monic quadratic

$$x^2 + bx + c = 0$$

admits two algebraically equivalent fixed-point representations:

$$x = -\frac{c}{b+x} \quad \text{and} \quad x = -b - \frac{c}{x},$$

each defining a Möbius (linear-fractional) transformation whose fixed points are the roots of the quadratic.

Iterating either form yields a continued fraction that converges to one of those roots:

$$x = -\frac{c}{b - \frac{c}{b - \frac{c}{b - \frac{c}{b - \frac{c}{b - \frac{c}{\ddots}}}}}}} \quad x = -b - \frac{c}{-b - \frac{c}{-b - \frac{c}{-b - \frac{c}{-b - \frac{c}{\ddots}}}}}.$$

Each layer of the fraction applies the same three-step Möbius action—an affine shift (add b), followed by inversion (take the reciprocal), and then a rescaling (scale by $-c$). These maps compose into each other and share the same fixed points, revealing how the static Vieta invariants of the quadratic reappear as dynamical fixed points of iteration.

Vieta relations

In 1591, François Viète showed that the coefficients (a , b , c , and d) of the general monic quartic

$$x^4 + Ax^3 + Bx^2 + Cx + D = 0$$

with roots x_1 , x_2 , x_3 , and x_4 , are fixed by four symmetric combinations of those roots—its sum of roots, pairwise product sum of roots, triple product sum of roots, and its product of roots.⁴⁹

$$\sum_{i=1}^4 x_i = -A \quad \sum_{i<j}^4 x_i x_j = B \quad \sum_{i<j<k}^4 x_i x_j x_k = -C \quad \prod_{i=1}^4 x_i = D$$

These relations define the static invariants of the system—the algebraic constraints that any transformation of its roots must preserve.

Let us now apply these insights to $Q(t)$ and $P(x)$ to identify the continued-fraction structure and Vieta invariants embedded within them.

Continued fractions for $Q(t)$

Applying Euler's method to $Q(t)$ gives the two Möbius maps

$$F(t) = -\frac{2\pi}{2\pi + t} \quad \text{and} \quad G(t) = -2\pi - \frac{2\pi}{t}.$$

Their fixed-point equations $t_1 = F(t_1)$ and $t_2 = G(t_2)$ unfold into continued fractions:

$$t_1 = -\frac{2\pi}{2\pi - \frac{2\pi}{2\pi - \frac{2\pi}{2\pi - \frac{2\pi}{2\pi - \frac{2\pi}{\ddots}}}}} \quad t_2 = -2\pi - \frac{2\pi}{-2\pi - \frac{2\pi}{-2\pi - \frac{2\pi}{-2\pi - \frac{2\pi}{\ddots}}}.$$

Each iteration is a Möbius transformation; each fraction is the iterative expression of the same dynamic.

$$t = F(G(t)) = G(F(t)) \quad \text{for } t \neq 0, -2\pi$$

These two Möbius maps are inverses.

$$F \circ G = G \circ F = \text{id}.$$

Thus F and G are inverse Möbius transformations away from their poles. They also act as natural transforms between whole number divisions of 2π and rational numbers. (See Appendix B for more correspondences.)

$$F\left(-\frac{2\pi}{k}\right) = -\frac{k}{k-1} \quad G(-k) = -2\pi\left(1 - \frac{1}{k}\right) \quad k \neq 1, -2\pi$$

Möbius transformations can be represented by 2×2 matrices, and their continued-fraction iterations correspond to repeated matrix multiplication. Each layer of these continued fractions corresponds to multiplying by the same 2×2 matrix M

$$M_F = \begin{pmatrix} 0 & -2\pi \\ 1 & 2\pi \end{pmatrix} \quad \det M_F = 2\pi, \quad \text{tr } M_F = 2\pi$$

$$M_G = \begin{pmatrix} -2\pi & -2\pi \\ 1 & 0 \end{pmatrix} \quad \det M_G = 2\pi, \quad \text{tr } M_G = -2\pi$$

the convergents of which are the ratios of entries of M^n . These fixed points t_1 and t_2 are the attractors/repellers around which the continued fractions organize, corresponding to the eigenlines of the Möbius action of M .

$Q(t)$ invariants

Because $Q(t)$ governs the reduced structure of the partition equation, its invariants—its roots and Vieta relations—provide the first glimpse of how the full quartic organizes its symmetry.

Roots of $Q(t)$

$$t_1 = -\pi + \sqrt{\pi(\pi - 2)} \quad t_2 = -\pi - \sqrt{\pi(\pi - 2)}$$

$Q(t)$ Vieta relations

The roots of $Q(t)$ sum to -2π . Their product is 2π . And their quadrance (the sum of their squares) is $4\pi(\pi - 1)$.

$$t_1 + t_2 = -2\pi \quad t_1 t_2 = 2\pi \quad t_1^2 + t_2^2 = 4\pi(\pi - 1)$$

$$\sum_{i=1}^2 t_i = -2\pi \quad \prod_{i=1}^2 t_i = 2\pi \quad \sum_{i=1}^2 t_i^2 = 4\pi(\pi - 1)$$

Note that the sum and the negative product coincide: $\sum t_i = -\prod t_i$.

Resolvent cubic $R(y)$

The even quartic $P(x)$ decomposes into two quadratics (in $t = x^2$) governed by the resolvent cubic $R(y)$:

$$R(y) = y^3 - 2\pi y^2 - 8\pi y + 16\pi^2 = (y - 2\pi)(y^2 - 8\pi).$$

The three roots of the resolvent cubic— r_1 , r_2 , and r_3 —encode the three distinct pairings of the quartic's four roots.⁵⁰

$$\begin{aligned} r_1 &= 2\pi & r_2 &= -2\sqrt{2\pi} & r_3 &= 2\sqrt{2\pi} \\ r_1 &= x_1x_2 + x_3x_4 & r_2 &= x_1x_3 + x_2x_4 & r_3 &= x_1x_4 + x_2x_3 \end{aligned}$$

And the quartic's roots encode the quadratic's roots as follows:

$$x_1^2 = t_1 \quad x_2^2 = t_1 \quad x_3^2 = t_2 \quad x_4^2 = t_2$$

Therefore,

$$r_1 = -(t_1 + t_2) \quad r_2 = -2\sqrt{t_1 t_2} \quad r_3 = 2\sqrt{t_1 t_2}$$

Vieta relations for $R(y)$

The resolvent cubic satisfies:

$$r_1 + r_2 + r_3 = 2\pi \quad r_1r_2 + r_1r_3 + r_2r_3 = -8\pi \quad r_1 r_2 r_3 = -(4\pi)^2$$

$$r_1^2 + r_2^2 + r_3^2 = 4\pi(\pi + 4) \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{2\pi}$$

$$\sum_{i=1}^3 r_i = 2\pi \quad \sum_{i<j}^3 r_i r_j = -8\pi \quad \prod_{i=1}^3 r_i = -(4\pi)^2$$

$$\sum_{i=1}^3 r_i^2 = 4\pi(\pi + 4) \quad \sum_{i=1}^3 \frac{1}{r_i} = \frac{1}{2\pi} \quad \sum r_i \sum \frac{1}{r_i} = 1$$

These cubic invariants reveal how the quadratic structure of $Q(t)$ lifts into the full quartic structure, organizing the three possible pairwise couplings between its four roots.

Quartic invariants $P(x)$

Returning to the full quartic polynomial in x ,

$$P(x) = x^4 + 2\pi x^2 + 2\pi = 0,$$

every root in t becomes 2 roots in x , since $x^2 = t$ defines a *two-sheeted cover* of the t -plane. This geometric doubling reflects the same algebraic symmetry that Dirac interpreted physically as the particle-antiparticle pairing built into relativistic quantum mechanics.

Roots of $P(x)$

The quartic's four roots are:

$$\begin{aligned} x_1 &= i \sqrt{\pi - \sqrt{\pi(\pi - 2)}} & x_2 &= -i \sqrt{\pi - \sqrt{\pi(\pi - 2)}} \\ x_3 &= i \sqrt{\pi + \sqrt{\pi(\pi - 2)}} & x_4 &= -i \sqrt{\pi + \sqrt{\pi(\pi - 2)}} \end{aligned}$$

Vieta relations for $P(x)$

The Vieta relations for these roots are:

They sum to 0.

Their pairwise-product sum is 2π .

Their triple-product sum is 0.

Their product is 2π .

Their quadrance (the sum of their squares) is -4π .

And the sum of their square moduli is 4π .

$$\sum_{i=1}^4 x_i = 0$$

$$\prod_{i=1}^4 x_i = 2\pi$$

$$\sum_{i=1}^4 x_i^2 = -4\pi$$

$$\sum_{i<j<k}^4 x_i x_j x_k = 0$$

$$\sum_{i<j}^4 x_i x_j = 2\pi$$

$$\sum_{i=1}^4 |x_i|^2 = 4\pi$$

Notice: $\sum x_i = \sum_{i<j<k} x_i x_j x_k = 0$, and $\prod x_i = \sum_{i<j} x_i x_j = 2\pi$.

The hyperbolic partition equation produces roots whose symmetric combinations collapse into exceptionally simple expressions involving 0 , 2π , -4π ; exactly the kind of internal order one hopes to find when looking for a generative story behind measured structure.

Newton power sums

The generalized power sum formula for these roots is given by

$$S_{k+4} + 2\pi S_{k+2} + 2\pi S_k = 0 \qquad S_k = \sum_{i=1}^4 x_i^k$$

with seeds $S_0 = 4$, $S_1 = 0$, $S_2 = -4\pi$, $S_3 = 0$. All odd sums vanish $S_{2n+1} = 0$.

Projective invariants

For a monic quartic $x^4 + ax^3 + bx^2 + cx + d$, the binary-quartic invariants I and J remain stable (up to normalization) under linear changes of variable.

While Vieta's relations describe symmetric combinations of the roots, I and J are projective invariants—they capture how the quartic persists under projective transformation.⁵¹

From I and J one constructs the quadratic discriminant Δ_x , which vanishes if and only if the quartic has a multiple root, and the elliptic j -

invariant which classifies the elliptic curve $Y^2 = P(X)$ up to isomorphism when $\Delta_x \neq 0$.

$$\Delta_x = \frac{4I^3 - J^2}{3^3} \qquad j = \frac{4^4 I^3}{\Delta_x}$$

The projective invariants for the quadratic $Q(t)$, the cubic $R(y)$, and the even quartic $P(x)$ are:

$\Delta_t =$ quadratic discriminant of $Q(t)$	$4\pi (\pi - 2)$
$I =$ quartic invariant of $P(x)$	$4\pi (\pi + 6)$
$J =$ quartic invariant of $P(x)$	$(4\pi)^2 (18 - \pi)$
$\Delta_y =$ cubic discriminant of $R(y)$	$2 (4\pi)^3 (2\pi - 4)^2$
$\Delta_x =$ quartic discriminant of $P(x)$	$2 (4\pi)^3 (2\pi - 4)^2$
$j =$ j -invariant of curve $Y^2 = P(X)$	$32 \frac{(\pi + 6)^3}{(\pi - 2)^2}$
$c_r =$ cross ratio (Möbius invariant)	$\frac{\pi - \sqrt{2\pi}}{\pi + \sqrt{2\pi}}$

These invariants describe how the underlying hyperbolic structure maintains identity under projective transformation.

The full partition equation

Up to this point, we have studied the homogeneous form of the hyperbolic partition equation—set equal to zero. We now consider the non-zero case, introducing a finite magnitude a .⁵² This is the step that moves the equation from its symmetric core into the gapped form. It allows us to derive the invariants of the full system when $a \neq 0$.

$$F(x) = \frac{1}{x} + x + \frac{x^3}{2\pi} = a \qquad a \neq 0$$

Multiplying through by x yields the monic depressed quartic

$$x^4 + px^2 + qx + r = 0,$$

where $p = 2\pi$, $q = -2\pi a$, $r = 2\pi$, and hence,

$$T(x) = x^4 + 2\pi x^2 - 2\pi ax + 2\pi = 0.$$

Roots of $T(x)$

When $a = (i^i)^{-4\pi/8} - \frac{m_p}{kg}$, the roots of this quartic system are: \mathfrak{K}_1 , \mathfrak{K}_2 , \mathfrak{K}_3 , and \mathfrak{K}_4 .

$$\mathfrak{K}_1 = 0.0854245431533304 \dots$$

$$\mathfrak{K}_2 = 3.66756753485501 \dots$$

$$\mathfrak{K}_3 = -1.87649603900417 \dots + 4.06615262615972 \dots i$$

$$\mathfrak{K}_4 = -1.87649603900417 \dots - 4.06615262615972 \dots i.$$

Where $\mathfrak{K}_1^2 =$ the fine-structure constant, $\mathfrak{K}_3 = \mathfrak{K}_r e^{\mathfrak{K}\theta i}$, $\mathfrak{K}_4 = \mathfrak{K}_r e^{-\mathfrak{K}\theta i}$, and $\mathfrak{K}_3\mathfrak{K}_4 = \mathfrak{K}_r^2$.

These roots form a structured quadratic-composition system obeying the Brahmagupta-Fibonacci identity—the rule that the product of two sums of squares is itself a sum of squares:⁵³

$$(A^2 + B^2)(C^2 + D^2) = (AC - BD)^2 + (AD + BC)^2 = \mathfrak{K}_r^4,$$

$$(A^2 + B^2)(C^2 + D^2) = (AC + BD)^2 + (AD - BC)^2 = \mathfrak{K}_r^4,$$

for any of the following consistent assignments of (A, B, C, D) :

$$A = Re(\mathfrak{K}_3) \quad B = Im(\mathfrak{K}_3) \quad C = Re(\mathfrak{K}_4) \quad D = Im(\mathfrak{K}_4)$$

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where $\mathfrak{K}_r^4 = (\mathfrak{K}_3\mathfrak{K}_4)^2$.

This identity expresses rotational invariance in the Euclidean plane: complex multiplication corresponds to a rotation and scaling that preserves the modulus \varkappa_r . It is the algebraic statement of metric coherence under transformation—the simplest realization of norm-preserving symmetry.

This quadratic identity generalizes hierarchically.

Euler showed that the product of two four-component sums of squares is again a four-square—the algebraic foundation of Hamilton’s quaternions, which describe rotations in 3-space.⁵⁴

Extended once more, Degen, Graves, and Cayley showed that the product of two eight-component sums of squares is again an eight-square—structurally defining the octonions, whose norm-preserving multiplication is deeply tied to exceptional symmetry and seven-dimensional geometry.

Each stage preserves the same invariant norm:

$$N(xy) = N(x)N(y),$$

where N is the quadratic norm.

Thus, $\mathbb{R} \rightarrow \mathbb{C} \rightarrow \mathbb{H} \rightarrow \mathbb{O}$ forms the complete chain of norm-preserving quadratic composition algebras, each doubling the dimension—and the internal symmetry—of the one before it.⁵⁵

Vieta relations for $T(x)$

The Vieta relations of the roots of $T(x)$ are: They sum to 0. Their product = 2π . Their pairwise-product sum = 2π . Their triple-product sum = $2\pi a$. Their quadrance (sum of their squares) = -4π . The sum of their cubes = $6\pi a$. The sum of their fourth powers = $8\pi(\pi - 1)$.

$$\varkappa_1 + \varkappa_2 + \varkappa_3 + \varkappa_4 = 0$$

$$\varkappa_1 \varkappa_2 \varkappa_3 \varkappa_4 = 2\pi$$

$$\varkappa_1 \varkappa_2 + \varkappa_1 \varkappa_3 + \varkappa_1 \varkappa_4 + \varkappa_2 \varkappa_3 + \varkappa_2 \varkappa_4 + \varkappa_3 \varkappa_4 = 2\pi$$

$$\varkappa_1 \varkappa_2 \varkappa_3 + \varkappa_1 \varkappa_2 \varkappa_4 + \varkappa_1 \varkappa_3 \varkappa_4 + \varkappa_2 \varkappa_3 \varkappa_4 = 2\pi a$$

$$\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2 + \varkappa_4^2 = -4\pi$$

$$\mathfrak{K}_1^3 + \mathfrak{K}_2^3 + \mathfrak{K}_3^3 + \mathfrak{K}_4^3 = 6\pi a$$

$$\mathfrak{K}_1^4 + \mathfrak{K}_2^4 + \mathfrak{K}_3^4 + \mathfrak{K}_4^4 = 8\pi (\pi - 1)$$

The reciprocal triple product sum of these roots = 0. Their reciprocal pairwise sum = 1. Their reciprocal sum = a . Their inverse bi-product sum = 1. Their inverse triple product sum = 0. Their square inverse biproduct sum = $1 + 1/\pi$. Their square inverse triple product sum = $-1/\pi$. The ratio of the 1D root sum to the 2D root sum = -1 . And their cross-ratio λ has a magnitude of 1, $|\lambda| = 1$, confirming that the system's Möbius invariants preserve metric symmetry.⁵⁶ (For the Vieta relations of $T'(x)$ see Appendix C.)

$$\frac{1}{\mathfrak{K}_1} + \frac{1}{\mathfrak{K}_2} + \frac{1}{\mathfrak{K}_3} + \frac{1}{\mathfrak{K}_4} = a$$

$$\frac{1}{\mathfrak{K}_1\mathfrak{K}_2} + \frac{1}{\mathfrak{K}_1\mathfrak{K}_3} + \frac{1}{\mathfrak{K}_1\mathfrak{K}_4} + \frac{1}{\mathfrak{K}_2\mathfrak{K}_3} + \frac{1}{\mathfrak{K}_2\mathfrak{K}_4} + \frac{1}{\mathfrak{K}_3\mathfrak{K}_4} = 1$$

$$\frac{1}{\mathfrak{K}_1\mathfrak{K}_2\mathfrak{K}_3} + \frac{1}{\mathfrak{K}_1\mathfrak{K}_2\mathfrak{K}_4} + \frac{1}{\mathfrak{K}_1\mathfrak{K}_3\mathfrak{K}_4} + \frac{1}{\mathfrak{K}_2\mathfrak{K}_3\mathfrak{K}_4} = 0$$

$$\frac{1}{\mathfrak{K}_1^2\mathfrak{K}_2^2} + \frac{1}{\mathfrak{K}_1^2\mathfrak{K}_3^2} + \frac{1}{\mathfrak{K}_1^2\mathfrak{K}_4^2} + \frac{1}{\mathfrak{K}_2^2\mathfrak{K}_3^2} + \frac{1}{\mathfrak{K}_2^2\mathfrak{K}_4^2} + \frac{1}{\mathfrak{K}_3^2\mathfrak{K}_4^2} = 1 + \frac{1}{\pi}$$

$$\frac{1}{\mathfrak{K}_1^2\mathfrak{K}_2^2\mathfrak{K}_3^2} + \frac{1}{\mathfrak{K}_1^2\mathfrak{K}_2^2\mathfrak{K}_4^2} + \frac{1}{\mathfrak{K}_1^2\mathfrak{K}_3^2\mathfrak{K}_4^2} + \frac{1}{\mathfrak{K}_2^2\mathfrak{K}_3^2\mathfrak{K}_4^2} = -\frac{1}{\pi}$$

$$\frac{\mathfrak{K}_1 + \mathfrak{K}_2}{\mathfrak{K}_3 + \mathfrak{K}_4} = -1 \quad \left| \frac{(\mathfrak{K}_1 - \mathfrak{K}_3)(\mathfrak{K}_2 - \mathfrak{K}_4)}{(\mathfrak{K}_1 - \mathfrak{K}_4)(\mathfrak{K}_2 - \mathfrak{K}_3)} \right| = 1$$

In product and summation notation these relations are:

$$\sum_{i=1}^4 \mathfrak{K}_i = 0$$

$$\sum_{i<j}^4 \mathfrak{K}_i \mathfrak{K}_j = 2\pi$$

$$\sum_{i<j<k}^4 \mathfrak{K}_i \mathfrak{K}_j \mathfrak{K}_k = 2\pi a$$

$$\prod_{i=1}^4 \mathfrak{K}_i = 2\pi$$

$$\sum_{i=1}^4 \mathfrak{K}_i^2 = -4\pi$$

$$\sum_{i=1}^4 \mathfrak{K}_i^3 = 6\pi a$$

$$\sum_{i=1}^4 \mathfrak{K}_i^4 = 8\pi (\pi - 1)$$

$$\sum_{i=1}^4 \frac{1}{\mathfrak{K}_i} = a$$

$$\sum_{i<j}^4 \frac{1}{\mathfrak{K}_i \mathfrak{K}_j} = 1$$

$$\sum_{i<j<k}^4 \frac{1}{\mathfrak{K}_i \mathfrak{K}_j \mathfrak{K}_k} = 0$$

$$\sum_{i<j}^4 \frac{1}{\mathfrak{K}_i^2 \mathfrak{K}_j^2} = 1 + \frac{1}{\pi}$$

$$\sum_{i<j<k}^4 \frac{1}{\mathfrak{K}_i^2 \mathfrak{K}_j^2 \mathfrak{K}_k^2} = -\frac{1}{\pi}$$

$$\frac{\sum 1\text{D roots}}{\sum 2\text{D roots}} = -1$$

$$|\lambda| = 1$$

$$\lambda = \frac{(\mathfrak{K}_1 - \mathfrak{K}_3)(\mathfrak{K}_2 - \mathfrak{K}_4)}{(\mathfrak{K}_1 - \mathfrak{K}_4)(\mathfrak{K}_2 - \mathfrak{K}_3)}$$

Notably, $\prod \mathfrak{K}_i = \sum_{i<j} \mathfrak{K}_i \mathfrak{K}_j$ remains independent of a , showing that a enters as a rank-one perturbation of the $a = 0$ factorization.

Newton power sums for $T(x)$

The Newton power sum formula for the roots of $T(x)$ is given by

$$S_k + p S_{k-2} + q S_{k-3} + r S_{k-4} = 0 \quad S_k = \sum_{i=1}^4 x_i^k.$$

Substituting $p = 2\pi$, $q = -2\pi a$, $r = 2\pi$ this becomes

$$S_k + 2\pi S_{k-2} - 2\pi a S_{k-3} + 2\pi S_{k-4} = 0$$

With seeds $S_0 = 4$, $S_1 = 0$, $S_2 = -4\pi$, and $S_3 = 6\pi a$.⁵⁷

From these recurrence relations follow a hierarchy of identities revealing how the partition parameter a propagates through successive power sums:

$$\mathfrak{K}_1^{-1} + \mathfrak{K}_2^{-1} + \mathfrak{K}_3^{-1} + \mathfrak{K}_4^{-1} = a$$

$$\mathfrak{K}_1^0 + \mathfrak{K}_2^0 + \mathfrak{K}_3^0 + \mathfrak{K}_4^0 = 4$$

$$\mathfrak{K}_1^1 + \mathfrak{K}_2^1 + \mathfrak{K}_3^1 + \mathfrak{K}_4^1 = 0$$

$$\mathfrak{K}_1^2 + \mathfrak{K}_2^2 + \mathfrak{K}_3^2 + \mathfrak{K}_4^2 = -4\pi$$

$$\mathfrak{K}_1^3 + \mathfrak{K}_2^3 + \mathfrak{K}_3^3 + \mathfrak{K}_4^3 = 3a (2\pi)$$

$$\mathfrak{K}_1^4 + \mathfrak{K}_2^4 + \mathfrak{K}_3^4 + \mathfrak{K}_4^4 = 4\pi (2\pi - 2)$$

$$\mathfrak{K}_1^5 + \mathfrak{K}_2^5 + \mathfrak{K}_3^5 + \mathfrak{K}_4^5 = -5a (2\pi)^2$$

$$\mathfrak{K}_1^6 + \mathfrak{K}_2^6 + \mathfrak{K}_3^6 + \mathfrak{K}_4^6 = (2\pi)^2 (3a^2 - 4\pi + 6)$$

$$\mathfrak{K}_1^7 + \mathfrak{K}_2^7 + \mathfrak{K}_3^7 + \mathfrak{K}_4^7 = 7a ((2\pi)^3 - (2\pi)^2)$$

$$\mathfrak{K}_1^8 + \mathfrak{K}_2^8 + \mathfrak{K}_3^8 + \mathfrak{K}_4^8 = (4\pi)^2 (2\pi^2 - 4\pi a^2 - 4\pi + 1)$$

Projective invariants of $T(x)$

The projective invariants for the quartic $T(x)$ are:

$$I = \text{quartic invariant of } T(x). \quad 4\pi (\pi + 6)$$

$$J = \text{quartic invariant of } T(x). \quad \pi^2 (288 - 18 \cdot 6a^2 - 16\pi)$$

$$\Delta_x = \text{quartic discriminant of } T(x). \quad \frac{4 I^3 - J^2}{3^3}$$

$$j(T) = j\text{-invariant of the elliptic curve } Y^2 = T(x).^{58} \quad \frac{4^4 I^3}{\Delta_x}$$

Together, these invariants characterize how the quartic persists under projective transformation, encoding the deeper geometric identity of the partition system.

The invariants of the hyperbolic partition equation—its Vieta relations, Möbius maps, reciprocal identities, power sums, and projective invariants—form a closed algebra under inversion and scaling by 2π . They function like algebraic conservation laws within the partition system: statements of what remains fixed as the roots transform, invert, and recombine.

The hyperbolic partition equation does more than approximate the fine-structure constant—it is a structured object with invariants, recurrences, reciprocal symmetries, Möbius dynamics, projective data, and norm-preserving identities. It gives us a disciplined object to test against the constants of Nature.

The ability to trace the structural invariants of this algebraic-geometric connection positions us to ask the next question: Can a world be built from Planck boundaries using just the binomial constructor and this partition logic?

With exact definitions for those boundaries, every constant of Nature could, in principle, be decomposed into precise geometric actions upon them—each becoming a faithful expression of how the hyperbolic structure partitions. This would allow us to test whether the partition logic of $T(x)$ —the logic that gives rise to the fine-structure constant—also governs the full set of 288 constants of Nature.