Part III: Geometry Speaks Physics

> “Behind it all is surely an idea so simple, so beautiful, that when we grasp it—in a decade, a century, or a millennium—we will all say to each other, how could it have been otherwise?”
> —John Archibald Wheeler[60]

Chapter 9: The Geometry Behind the Constants

For centuries, the constants of Nature have stood as a list of empirical facts without a structural origin story. This book treats that data set as a connected catalog with internal grammar. With candidate closed forms for the Planck boundaries and a fixed partition structure, we can now examine the constants of Nature for a shared transform syntax.

This chapter has three tasks. First, it shows how the Transform Dictionary organizes constants into families rather than isolated facts. Second, it examines the three constants that allow the Dirac equation to describe relativistic quantum matter—$c, \hbar, m_e$—and shows that their stabilizing geometries point to the figure-eight knot complement. Third, it explains why the completed Dictionary has 288 entries with the same 8×36 partition structure that appears in the Cayley-Menger reconstruction of tetrahedral volume and in the conjugate dilogarithm measure of the figure-eight construction.

At the back of this book, you will find a map called the *Transform Dictionary*. It presents all 288 constants of Nature, each decoded using just two structural rules: the binomial constructor and the hyperbolic partition equation. Together, these rules convert every constant into a coherent 2-part Planck-bounded transform: an external boundary anchor paired with an internal geometric action. The resulting expressions reconstruct the magnitudes and dimensions of the constants while revealing a candidate structure responsible for the transformation each constant facilitates.

The Transform Dictionary is organized by partition structure. The first two columns collect polar transforms of the system’s roots: first 36 radial transforms, then 36 radius-bound transforms. The next six columns, with 36 equations each, define the linear Cartesian transforms that encode pairwise and triplewise interactions among those roots.

In Chapter 3, we introduced Nature’s energy architecture, corresponding to column 3 of the Transform Dictionary: the 2-part products. In the complete Dictionary, the rest of the constants of Nature organize into similar families. The same internal geometry repeats across an entire family, while the external boundary anchor changes from constant to constant.

For example, in column 6—the 2-part quadrances—14 constants are generated by the same quadrature transform, governed by the universal parabolic constant P_{up}. This constant is a companion quadrature to $4\pi/8$, and it functions as a control parameter for the transformation applied throughout this family.

$$\int_{-1}^{1} \sqrt{1 - x^2} = \frac{4\pi}{8} \qquad \int_{-1}^{1} \sqrt{1 + x^2} = P_{up}$$

$$\mathrm{kg} \vdots \frac{1}{\mathrm{m}} = \frac{1}{2\pi}\left(\frac{\mathrm{kg}}{l_p\, m_p}\right)\left(1 - \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\mathrm{A}_{\mathrm{mass}} \vdots \frac{1}{\mathrm{m}} = \frac{1}{2\pi}\left(\frac{\mathrm{A}_{\mathrm{mass}}}{l_p\, m_p}\right)\left(1 - \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\frac{1}{\mathrm{m}} \vdots \mathrm{kg} = 2\pi\left(\frac{l_p\, m_p}{\mathrm{m}}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\frac{1}{\mathrm{m}} \vdots \mathrm{A}_{\mathrm{mass}} = 2\pi\left(\frac{l_p\, m_p}{\mathrm{m}\, \mathrm{A}_{\mathrm{mass}}}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_C = 2\pi\left(\frac{l_p\, m_p}{m_e}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_\mu = 2\pi\left(\frac{l_p\, m_p}{m_\mu}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_+ = 2\pi\left(\frac{l_p\, m_p}{m_+}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_n = 2\pi\left(\frac{l_p\, m_p}{m_n}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_\tau = 2\pi\left(\frac{l_p\, m_p}{m_\tau}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_{-} = \left(\frac{l_p\, m_p}{m_e}\right)\left(1 + \frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2 - \text{ж}_4{}^2\right) \boxtimes\right)$$

$$\lambda_{\mu\_} = \left(\frac{l_p\, m_p}{m_\mu}\right)\left(1+\frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2-\text{ж}_4{}^2\right)\boxtimes\right)$$

$$\lambda_{+\_} = \left(\frac{l_p\, m_p}{m_+}\right)\left(1+\frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2-\text{ж}_4{}^2\right)\boxtimes\right)$$

$$\lambda_{n\_} = \left(\frac{l_p\, m_p}{m_n}\right)\left(1+\frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2-\text{ж}_4{}^2\right)\boxtimes\right)$$

$$\lambda_{\tau\_} = \left(\frac{l_p\, m_p}{m_\tau}\right)\left(1+\frac{P_{up}\, i}{8}\left(\text{ж}_3{}^2-\text{ж}_4{}^2\right)\boxtimes\right)$$

Notation note: In the Transform Dictionary, the symbol ⁝ is a relationship marker in the name of a constant. For example, $\text{kg}\,\text{⁝}\,1/\text{m}$ means "the kilogram-inverse meter relationship," and $\mathrm{A}_{\mathrm{mass}}\,\text{⁝}\,1/\text{m}$ means "the atomic mass unit-inverse meter relationship." It is not an arithmetic operation; it names the relation being decoded.

Each of these fourteen constants uses the same internal quadrature: a parabolic deformation controlled by P_{up}, acting on a 2-part quadrance of the hyperbolic roots to invert about $\boxtimes$. What varies from constant to constant is not their internal geometry, but the external boundary anchor.

By contrast, in column 5—the 2-part sums—a group of 19 constants of Nature organize through lemniscate division, governed by the lemniscate constant L, using a 2-part difference of roots.

$$\mathrm{A}_{\mathrm{mass}} = \sqrt{\frac{18}{32}\,\frac{P_{up}}{\text{ж}_1{}^6}}\,(m_e)\left(1+\frac{1}{L}\left(\text{ж}_1-\text{ж}_2\right)\boxtimes\right)$$

$$\text{kg}\,\text{⁝}\,\mathrm{A}_{\mathrm{mass}} = \sqrt{\frac{32}{18}\,\frac{\text{ж}_1{}^6}{P_{up}}}\left(\frac{\text{kg}}{m_e}\right)\left(1-\frac{1}{L}\left(\text{ж}_1-\text{ж}_2\right)\boxtimes\right)$$

$$\mathrm{A}_{\mathrm{r}}(\mathrm{e}) = \sqrt{\frac{32}{18}\,\frac{\text{ж}_1{}^6}{P_{up}}}\left(1-\frac{1}{L}\left(\text{ж}_1-\text{ж}_2\right)\boxtimes\right)$$

$$\mathrm{A}_{\mathrm{r}}(+) = \sqrt{\frac{32}{18}\,\frac{\text{ж}_1{}^6}{P_{up}}}\left(\frac{m_+}{m_e}\right)\left(1-\frac{1}{L}\left(\text{ж}_1-\text{ж}_2\right)\boxtimes\right)$$

$$\mathrm{A_r(n)} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_n}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\mathrm{A_r(de)} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_{\mathrm{de}}}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\mathrm{A_r(he)} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_{\mathrm{he}}}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\mathrm{A_r(tri)} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_{\mathrm{tri}}}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\mathrm{A_r}(\alpha) = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_\alpha}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_e}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_\Delta}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_n-m_+}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_\mu}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_\mu}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_+}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_+}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_n}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_n}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_\tau}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_\tau}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_{\mathrm{de}}}{\mathrm{A_{mass}}} = \sqrt{\frac{32}{18}\,\frac{ж_1{}^6}{P_{up}}}\left(\frac{m_{\mathrm{de}}}{m_e}\right)\left(1-\frac{1}{L}\left(ж_1-ж_2\right)\boxtimes\right)$$

$$\frac{m_{\text{he}}}{\mathrm{A}_{\text{mass}}} = \sqrt{\frac{32}{18}\,\frac{\text{ж}_1{}^6}{P_{up}}}\left(\frac{m_{\text{he}}}{m_e}\right)\left(1-\frac{1}{L}(\,\text{ж}_1-\text{ж}_2\,)\boxtimes\right)$$

$$\frac{m_{\text{tri}}}{\mathrm{A}_{\text{mass}}} = \sqrt{\frac{32}{18}\,\frac{\text{ж}_1{}^6}{P_{up}}}\left(\frac{m_{\text{tri}}}{m_e}\right)\left(1-\frac{1}{L}(\,\text{ж}_1-\text{ж}_2\,)\boxtimes\right)$$

$$\frac{m_{\alpha}}{\mathrm{A}_{\text{mass}}} = \sqrt{\frac{32}{18}\,\frac{\text{ж}_1{}^6}{P_{up}}}\left(\frac{m_{\alpha}}{m_e}\right)\left(1-\frac{1}{L}(\,\text{ж}_1-\text{ж}_2\,)\boxtimes\right)$$

Together, these groups illustrate how entire families of constants are anchored to specific root transforms within the system. A list becomes a language when its entries obey a shared syntax. This is the point of the Transform Dictionary: to expose a grammar of relations that can be checked across the whole dataset.

To begin exploring the content of the Transform Dictionary, we turn to the three constants that allow the Dirac equation to describe relativistic quantum matter: c, $\hbar$, and m_e. These constants form the interface between algebra, geometry, and physics. They allow Dirac's formalism to combine quantum mechanics and special relativity into a single coherent framework.

In the Dictionary, every constant $\mathcal{C}$ occupies the same general form,

$$\mathcal{C} = \mathrm{A}_{\text{ext}}\,\mathrm{B}_{\text{ext}}\left(1+\mathrm{A}_{\text{int}}\boxtimes\right),$$

but each uses a different internal stabilizer. In this construction, the speed of light isolates the Gieseking volume scale. The reduced Planck constant isolates the real modular partition scale. The electron mass isolates the paired orientable hyperbolic volume of the figure-eight knot complement. The question is whether these three stabilizers are unrelated, or whether they belong to one geometric source.

The Transform Dictionary entry for the speed of light in vacuum externally constructs c from one Planck length divided by one Planck time. Its internal action uses the root difference $(\,\text{ж}_1-\text{ж}_2\,)$ to internally invert about $\boxtimes$ with a magnitude of the square root of Gieseking's constant $\sqrt{G_{\text{Gi}}}$. This construction characterizes the coherent limit of velocity and ties that limit to the volume scale of the minimal non-orientable cusped hyperbolic 3-manifold.

speed of light in vacuum

$$c = \left(\frac{l_p}{t_p} \right) \left(1 + \sqrt{G_{\mathrm{Gi}}} \left(\text{ж}_1 - \text{ж}_2 \right) \boxtimes \right)$$

Where l_p = the Planck length, t_p = the Planck time, G_{Gi} = Gieseking's constant, ж_1 = the 1st hyperbolic partition constant, ж_2 = the 2nd hyperbolic partition constant, and $\boxtimes$ = the hyperbolic inversion boundary.

$c = 2.99792458097898 \ldots \times 10^{8}\ \mathrm{m/s}$ prediction
$c = 2.99792458 \times 10^{8}\ \mathrm{m/s}$ CODATA 2022, 9-digit match
$c = 2.99792458 \times 10^{8}\ \mathrm{m/s}$ CODATA 2018, 9-digit match
$\Delta_{\mathrm{precision}} = 0.00$
$\Delta_{\mathrm{scaled}} = 0.00$

Also listed as the *natural unit of velocity*.

This construction characterizes the coherent limit of velocity and points to a hyperbolic volume scale associated with the minimal non-orientable cusped 3-manifold.

reduced Planck constant

$$\hbar = \left(\frac{{l_p}^2 \, m_p}{t_p} \right) \left(1 + \sqrt{\zeta(2)} \, {\text{ж}_\theta}^2 \boxtimes \right)$$

Where l_p = the Planck length, m_p = the Planck mass, t_p = the Planck time, $\zeta(x)$ = the Riemann zeta function, ж_θ = the hyperbolic radian constant, and $\boxtimes$ = the hyperbolic inversion boundary.

$\hbar = 1.05457181759658 \ldots \times 10^{-34}\ \mathrm{J\,s}$ prediction
$\hbar = 1.054571817 \times 10^{-34}\ \mathrm{J\,s}$ CODATA 2022, 10-digit match
$\hbar = 1.054571817 \times 10^{-34}\ \mathrm{J\,s}$ CODATA 2018, 10-digit match
$\Delta_{\mathrm{precision}} = 0.00$
$\Delta_{\mathrm{scaled}} = 0.00$

Also listed as the *atomic unit of action*, and the *natural unit of action*.

The Transform Dictionary entry for the reduced Planck constant defines a bounded action involving one Planck length squared, multiplied by one Planck mass, divided by one Planck time. Its internal stabilizer is a polar transform, inverting about $\boxtimes$ through $\sqrt{\zeta(2)}$. This transformation characterizes the atomic unit of action in exact agreement with the listed value—and belongs to a family of 14 related polar transforms.

$$\Delta_{\nu_{\mathrm{Cs}}} = \frac{E_1}{2\pi}\left(\frac{t_p}{{l_p}^2\, m_p}\right)\left(1 - \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\mathrm{eV} \vdots \mathrm{Hz} = \frac{\mathrm{eV}}{2\pi}\left(\frac{t_p}{{l_p}^2\, m_p}\right)\left(1 - \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\mathrm{J} \vdots \mathrm{Hz} = \frac{\mathrm{joule}}{2\pi}\left(\frac{t_p}{{l_p}^2\, m_p}\right)\left(1 - \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\mathrm{Hz} \vdots \mathrm{J} = 2\pi\left(\frac{{l_p}^2\, m_p}{\mathrm{s}\, t_p}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\mathrm{Hz} \vdots \mathrm{eV} = 2\pi\ \frac{\mathrm{joule}}{\mathrm{eV}}\left(\frac{{l_p}^2\, m_p}{\mathrm{s}\, t_p}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\dot{\hbar} = \frac{\mathrm{joule}}{\mathrm{eV}}\left(\frac{{l_p}^2\, m_p}{t_p}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\hbar = \left(\frac{{l_p}^2\, m_p}{t_p}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$h = 2\pi\left(\frac{{l_p}^2\, m_p}{t_p}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$q_c = \pi\left(\frac{{l_p}^2\, m_p}{t_p\, m_e}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$2q_c = 2\pi\left(\frac{{l_p}^2\, m_p}{t_p\, m_e}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\frac{\mu_B}{e} = \frac{1}{2}\left(\frac{{l_p}^2\, \mathrm{C}\, m_p}{t_p\, m_e}\right)\left(1 + \sqrt{\zeta(2)}\ {\text{ж}_\theta}^2\ \boxtimes\right)$$

$$\frac{\mu_N}{e} = \frac{1}{2}\left(\frac{{l_p}^2\,\mathrm{C}\,m_p}{t_p\,m_+}\right)\left(1+\sqrt{\zeta(2)}\,{ж_\theta}^2\,\boxtimes\right)$$

$${G_0}^{-1} = \frac{\pi}{{ж_1}^2}\left(\frac{{l_p}^2 m_p}{t_p\,{q_p}^2}\right)\left(1+\sqrt{\zeta(3)}\,{ж_\theta}^2\,\boxtimes\right)$$

$$G_0 = \frac{{ж_1}^2}{\pi}\left(\frac{t_p\,{q_p}^2}{{l_p}^2 m_p}\right)\left(1-\sqrt{\zeta(3)}\,{ж_\theta}^2\,\boxtimes\right)$$

The appearance of $\zeta(3)$ instead of $\zeta(2)$ in the last two constants signals that the inverse quantum of conductance ${G_0}^{-1}$ and the quantum of conductance G_0 probe a deeper layer of the zeta function's structure than the twelve constants that precede them.[61]

electron mass

$$m_e = 2\mathrm{V}_{\mathrm{fe}}\left(\frac{{m_p}^4}{\mathrm{kg}^3}\right)\left(1+\left(\frac{14}{4s}\right){ж_r}^4\,\boxtimes\right)$$

Where V_{fe} = the figure-eight knot hyperbolic volume, m_p = the Planck mass, s = the arc length of the unit lemniscate, $ж_r$ = the hyperbolic radius constant, and $\boxtimes$ = the hyperbolic inversion boundary.

$m_e = 9.10938371013637\ldots \times 10^{-31}$ kg	prediction
$m_e = 9.1093837139(28) \times 10^{-31}$ kg	CODATA 2022, $\sigma = -1.34$
$m_e = 9.1093837015(28) \times 10^{-31}$ kg	CODATA 2018, $\sigma = +3.08$
	$\Delta_{\mathrm{precision}} = 0.00$
	$\Delta_{\mathrm{scaled}} = 4.42$

Also listed as the *atomic unit of mass*, and the *natural unit of mass*.

The Transform Dictionary entry for the electron mass constructs m_e from Planck mass boundaries normalized by the cubic kilogram. The external geometric coefficient is $2\mathrm{V}_{\mathrm{fe}}$, the volume of two figure-eight knot complements. $2G_{\mathrm{Gi}} = \mathrm{V}_{\mathrm{fe}}$.

Thus, the electron-mass transform points to four Gieseking volume units, paired into two orientable figure-eight volumes. The figure-eight knot complement and its sister manifold share the minimal-volume role among orientable cusped hyperbolic 3-manifolds; the Gieseking manifold supplies the corresponding minimal non-orientable volume scale. In this entry, the electron mass is anchored to the paired orientable layer.[62] Internally this configuration is stabilized by quartic radial transform $ж_r{}^4$, inverting across ⊠ with a coefficient that divides 14 parts by 4 arc lengths of the unit lemniscate.

This transform characterizes the electron mass and gives a value within 1.34 σ of the CODATA 2022 value.

These three entries place three hyperbolic invariants in the stabilizer slot: $\sqrt{G_{\mathrm{Gi}}}$, $\sqrt{\zeta(2)}$, and $2\mathrm{V}_{\mathrm{fe}}$. The striking fact is that these three values—appearing in the constants that calibrate Dirac's equation—can be read as different facets of one underlying construction. The speed of light isolates the Gieseking volume scale. The reduced Planck constant isolates the real modular partition scale. The electron mass isolates the paired hyperbolic volume of the figure-eight knot complement. Let us examine how.

A canonical expression for the hyperbolic volume of the figure-eight knot complement V_{fe} is:

$$\mathrm{V}_{\mathrm{fe}} = i\left[Li_2\left(\frac{1}{\varphi_i}\right) - Li_2(\varphi_i)\right]$$

Where $Li_2(x)$ = the dilogarithm, $i = (-1)^{1/2} = e^{i\pi/2}$ the imaginary unit, and the imaginary golden ratio $= \varphi_i = (-1)^{1/3} = e^{i\pi/3}$, principal branch.

Each dilogarithm naturally decomposes into real and imaginary parts. The real parts encode a spherical partition through $\Gamma(5)$, while the imaginary parts correspond to ± the volume of the Gieseking volume.[63]

$$Li_2\left(\frac{1}{\varphi_i}\right) = \left(\frac{4\pi}{\Gamma(5)}\right)^2 - G_{\mathrm{Gi}}\,i \qquad Li_2(\varphi_i) = \left(\frac{4\pi}{\Gamma(5)}\right)^2 + G_{\mathrm{Gi}}\,i$$

Where $\Gamma(x)$ = the gamma function, $\Gamma(5) = 24$, and G_{Gi} = Gieseking's constant.[64]

From this decomposition, the unity of the three Dirac constants becomes explicit. The speed of light uses the square root of the imaginary component $\sqrt{G_{\mathrm{Gi}}}$. The reduced Planck constant uses the real modular partition scale $\sqrt{6}\left(\frac{4\pi}{\Gamma(5)}\right) = \sqrt{\zeta(2)}$. And the electron mass uses the paired orientable hyperbolic volume $4G_{\mathrm{Gi}} = 2\mathrm{V}_{\mathrm{fe}}$.

Each constant c, $\hbar$, m_e isolates a distinct facet of the same canonical construction: the imaginary component, the real modular component, and the total orientable hyperbolic volume. Together, they single out the figure-eight knot complement as the minimal orientable cusped hyperbolic 3-manifold capable of supporting the three stabilizing geometries that appear in Dirac's equation. The figure-eight knot complement is not just another object added to the list. It is the global shape that lets the three entries be read together. The speed of light, the reduced Planck constant, and the electron mass become different expressions of one coherent geometric source.

The figure-eight knot complement is not just any manifold. It is a minimal hyperbolic 3-manifold compatible with a self-consistent partition structure.[65] Among infinite possibilities, Nature appears to be using the minimal sufficient stage.

This is where the language of topological persistence provides a useful analogy. In familiar field-theoretic examples—a vortex, a Skyrmion, or a Hopfion—an object can persist because the field has taken on a globally nontrivial form. Its identity is carried by the whole arrangement. It cannot simply relax to the empty state unless the field crosses a singular transition, breaks the configuration, or meets an opposite winding.

The point here is similar but more general: geometry can carry object-like identity. A persistent thing can be a conserved form of organization, rather than a little piece of material. In this chapter, the figure-eight knot complement plays that clarifying role. It is the global arena in which the transformations associated with c, $\hbar$ and m_e can be understood as facets of one persistent structure.

Let us look even closer at the character of that arena.

288 Invariant Reparameterizations

To see where the number 288 enters, we begin with the tetrahedral unit of hyperbolic volume. The figure-eight knot complement decomposes into two ideal tetrahedra, so the tetrahedron is the natural geometric unit of the construction.

For any tetrahedron with edge lengths d_{ij}, where the vertices are i, $j \in \{1,2,3,4\}$, the squared volume is given by the Cayley-Menger formula:

$$V^2 = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & {d_{12}}^2 & {d_{13}}^2 & {d_{14}}^2 \\ 1 & {d_{21}}^2 & 0 & {d_{23}}^2 & {d_{24}}^2 \\ 1 & {d_{31}}^2 & {d_{32}}^2 & 0 & {d_{34}}^2 \\ 1 & {d_{41}}^2 & {d_{42}}^2 & {d_{43}}^2 & 0 \end{vmatrix},$$

Here $d_{ij} = d_{ji}$ is the edge length between vertices i and j. The determinant is built entirely from squared pairwise distances; the Cayley-Menger formula reconstructs volume from distances alone.

The factor $1/288$ in this equation is the tetrahedral case of the general Cayley-Menger coefficient for an n-simplex.

$${V_n}^2 = \frac{(-1)^{n+1}}{2^n\,(n!)^2}\det(\mathrm{CM})$$

An n-simplex is the convex hull of $(n+1)$ points in general position. For a tetrahedron, $n = 3$, so the coefficient becomes

$$\frac{1}{2^3\,(3!)^2} = \frac{1}{288}.$$

Thus, the Cayley-Menger normalization naturally separates into two factors 8×36. The factor 8 comes from the metric conversion between squared distances and inner products. The Cayley-Menger determinant[66] begins with squared distances between vertices. To recover volume, those distances must be converted into the inner-product structure of three independent edge directions—that is, into the Gram matrix of the tetrahedron's edge vectors. In the general Cayley-Menger coefficient, this distance-to-inner-product conversion appears as the factor 2^n. For a tetrahedron, $n = 3$, so the metric factor is $2^3 = 8$.

The factor $(3!)^2 = 36$ comes from the simplex-to-parallelepiped conversion. Choose one vertex of a tetrahedron and draw the three edge vectors extending from it. These vectors span a three-dimensional parallelepiped. A tetrahedron occupies only 1/3! of that parallelepiped volume. Since the Cayley-Menger formula gives squared volume, this contribution is squared.

Thus the 288 decomposes into a metric factor and a simplex-volume factor. The 8 records the three metric polarization steps needed to pass from squared distances to inner products. The 36 records the squared simplex-to-parallelepiped conversion.

This gives a natural 8 × 36 decomposition of tetrahedral volume normalization. In the Transform Dictionary, this same decomposition becomes an organizing hypothesis: eight metric polarization classes paired with thirty-six ordering states. The same arithmetic that reconstructs tetrahedral volume from pairwise distances also organizes the Transform Dictionary: eight columns of metric polarization, each containing thirty-six ordering states.

The Conjugate Measure

The same 8 × 36 split appears when the real parts of the figure-eight dilogarithm pair are added. The conjugate sum factors into the product of one unit sphere divided by 8 and another divided by 36:

$$\left(\frac{4\pi}{8}\right)\left(\frac{4\pi}{36}\right) = \left[Li_2\left(\frac{1}{\varphi_i}\right) + Li_2(\,\varphi_i\,)\right].$$

Here $\pi =$ Archimedes' constant, $Li_2(x) =$ the polylogarithm of order 2 of x, and $\varphi_i = (-1)^{1/3} =$ the imaginary golden ratio.

Thus, the same two factors that normalize tetrahedral volume also partition the conjugate measure of the figure-eight construction. The same split organizes the Transform Dictionary—8 columns with 36 entries each—corresponding to eight polarization degrees paired with thirty-six ordering states.

This is the proposed symmetry architecture organizing all 288 constants of Nature. The arithmetic that normalizes a tetrahedron's volume is the same arithmetic that organizes the Transform Dictionary. One geometry. One partition logic. One ledger of conversions.

The eight columns correspond to metric polarization classes. The thirty-six entries in each column correspond to the ordering states available with that polarization class. The same 8 × 36 structure appears both in tetrahedral volume reconstruction and in the conjugate measure of the figure-eight dilogarithm pair.

We uncovered this structure using two constructive rules: the binomial constructor, which governs how external and internal layers coherently combine, and the hyperbolic partition equation, which governs how those layers coherently partition.

The binomial constructor was a rule we observed by painstakingly examining the combinatorial structure of the constants of Nature. We did not yet know the reason for this rule, but we observed it to be present—so we encoded it as a constructive constraint, and then asked what kind of geometry could make that rule natural.

The hyperbolic partition equation arose by modifying de Vries's approximation for the fine-structure constant[67] with a normalized Planck mass gap. This revealed a geometric structure with an explosion of connective logic—logic we used to decode the constants of Nature as geometric transforms about Planck boundaries.

But why were these two rules there?

Now that we've identified the root geometry of the system, we can ask whether those two rules are structural consequences of it. Are these two rules independent constructions—or consequences of the figure-eight knot complement's own architecture? If they arise from that architecture, then the framework becomes self-sourcing: the same geometry that organizes the constants also supplies the rules by which their transformations persist.

In the next chapter, we ask whether the binomial constructor and the hyperbolic partition equation are independent assumptions or consequences of the figure-eight knot complement itself. There we test whether both can be understood through the knot's double-cover architecture. From there, we step outward to examine the unique transform space compatible with that knot—a 24-dimensional geometry whose Leech-lattice structure may quantize all Planck-bounded transitions and fix the number of distinct link-states available between minimal persistent arenas.