
¹ See Merlin Donald, *Origins of the Modern Mind: Three Stages in the Evolution of Culture and Cognition* (Harvard University Press, 1991), for a widely respected analysis of how mythic culture structured early human cognition.

² Cf. Albert Einstein's discussion of the aim of physics in *Ideas and Opinions* (Crown, 1954), especially the section on the unity of natural law.

³ Thales, often called Thales the Measurer, was famous for applying measurement ideas—for example, using similar triangles to estimate the height of pyramids by comparing shadows. He is credited with *using* measurement in clever ways. However, Thales did not create a system of measurement in the sense of setting down universal, standardized methods. He used measurement for practical purposes—problem solving, surveying—but we have no record that he invented a broader conceptual framework for measurement itself.

Anaximander, who was either a student of Thales or his intellectual heir, is credited with inventing systematic methods—meaning he tried to systematize how time and space could be measured as part of a consistent, ongoing framework—crucial for science and cosmology. He linked measures of time to measures of space.

That's why historians say that Anaximander invented systematic methods of measurement, even though Thales was already doing brilliant practical measuring.

⁴ A. G. D. van der Waerden, *Science Awakening*, 1961, pp. 87–90; Dirk L. Couprie, *Heaven and Earth in Ancient Greek Cosmology*, 2011 — both discuss Anaximander's introduction of the gnomon.

⁵ See Mark Schiefsky (ed.), *Hippocratic Writings and Early Greek Science*, Harvard Center for Hellenic Studies (2013); also Andrew Gregory, *Anaximander: A Re-assessment*, Bloomsbury (2008), chapter 3.

⁶ See Aristotle, *Physics* III.5 (204b–205a) for the report on Anaximander's apeiron and the principle of reciprocal justice; also S. K. L. Maharaj, *Early Greek Theories of Nature*, 2018.

⁷ Aristotle, *Physics* II.8; *Metaphysics* XII; and Jonathan Barnes, *Aristotle: A Very Short Introduction*, 2000.

⁸ Galileo Galilei, *Discourses and Mathematical Demonstrations Relating to Two New Sciences* (1638), First Day.

⁹ Galileo, *Two New Sciences* (1638), Proposition I, Theorems I–III; see also Stillman Drake's commentary in his 1978 translation.

¹⁰ Galileo Galilei, 1632. Dialogue Concerning the Two Chief World Systems. Translated by Stillman Drake, p. 186; Walter Isaacson, *Einstein*, pp. 108–9.

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- ¹¹ See Einstein, *Relativity: The Special and General Theory* (1916); also Howard & Norton, “The Hole Argument” in *The Stanford Encyclopedia of Philosophy* for connections between Galilean and Einsteinian relativity.
- ¹² For the conceptual lineage from classical invariance to modern symmetry principles, see Hermann Weyl, *Symmetry* (1952); and Noether’s original 1918 paper, “Invariante Variationsprobleme.”
- ¹³ Johannes Kepler, *Astronomia Nova* (1609); *Harmonices Mundi* (1619).
- ¹⁴ See Richard Westfall, *Never at Rest: A Biography of Isaac Newton*, Cambridge University Press (1980), chapter 4; also Curtis Wilson, “From Kepler’s Laws to Newton’s Law,” *Archive for History of Exact Sciences* (1989).
- ¹⁵ Richard Westfall, *Never at Rest: A Biography of Isaac Newton*, Cambridge University Press (1980); especially chapters 4–7 for Newton’s synthesis of Galileo and Kepler.
- ¹⁶ See Isaac Newton, *Opticks* (1704); also Westfall (1980), chapters 14–16 for Newton’s experimental programs.
- ¹⁷ Newton, “De Analysisi” (1669) and “Method of Fluxions” (1671); see Niccolò Guicciardini, *Isaac Newton on Mathematical Certainty and Method*, MIT Press (2009).
- ¹⁸ Arthur Cayley, “A Memoir on the Theory of Matrices,” *Philosophical Transactions of the Royal Society* (1858); James Joseph Sylvester, “On the Theory of Linear Transformations,” *Cambridge and Dublin Mathematical Journal* (1851).
- ¹⁹ Cayley (1858), *ibid.* This is the historical origin of the modern matrix concept.
- ²⁰ William Rowan Hamilton, *Elements of Quaternions* (1866); see also Thomas Hankins, *Sir William Rowan Hamilton*, Johns Hopkins (1980).
- ²¹ Bernhard Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (Habilitationvortrag, 1854). English translation: “On the Hypotheses Which Lie at the Foundations of Geometry.”
- ²² See Jeremy Gray, *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic*, Oxford University Press (1989), chapter 10.
- ²³ Albert Einstein, “The Foundation of the General Theory of Relativity,” *Annalen der Physik* (1916). See also Abraham Pais, *Subtle is the Lord* (1982).
- ²⁴ Einstein (1916), *ibid.* For experimental confirmations, see Will, C. M., *Theory and Experiment in Gravitational Physics* (1993).
- ²⁵ Paul A. M. Dirac, “The Quantum Theory of the Electron,” *Proceedings of the Royal Society A* (1928).

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- ²⁶ Paul Dirac, *The Principles of Quantum Mechanics*, 1930; see also Dirac’s “Pretty mathematics” quote in Graham Farmelo, *The Strangest Man: The Hidden Life of Paul Dirac* (2009).
- ²⁷ Dirac’s fascination with α is well documented in his interviews and correspondence; see P. A. M. Dirac, *Directions in Physics* (1978), Lecture 1.
- ²⁸ William Thurston, *Three-Dimensional Geometry and Topology*, Vol. 1 (Princeton University Press, 1997).
- ²⁹ Grigori Perelman, “The Entropy Formula for the Ricci Flow and its Geometric Applications,” arXiv:math/0211159 (2002); see also John Morgan & Gang Tian, *Ricci Flow and the Poincaré Conjecture* (2007).
- ³⁰ C. Adams, *The Knot Book*, American Mathematical Society (2004), chapter 7; and Thurston (1997), *ibid.*
- ³¹ Minkowski space is the four-dimensional geometric arena of special relativity, where time and space are interwoven by a metric that preserves causality. This causal structure, defined by the light cone—the hyperbolic double cone—gives the Dirac equation its form: evolution within this space must preserve coherence across time-like separations while allowing lawful transformation across space-like ones.
- For standard treatments of Minkowski geometry, causality, and the role of the light cone in special relativity, see H. Minkowski, “Space and Time” (1908); W. Rindler, *Introduction to Special Relativity* (Oxford University Press, 1991); and R. Penrose, *The Road to Reality* (Vintage, 2004), chapters 16–18.
- ³² Emmy Noether, “Invariante Variationsprobleme,” *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, Math-Phys. Klasse (1918).
- ³³ See Katherine Brading & Harvey R. Brown, “Noether’s Theorems and Gauge Symmetries,” in *The Stanford Encyclopedia of Philosophy* (2022), for a modern conceptual overview.
- ³⁴ William Thomson (Lord Kelvin), “*Electrical Units of Measurement*,” lecture to the Institution of Civil Engineers, London, 3 May 1883; reprinted in *Popular Lectures and Addresses*, Vol. I (London: Macmillan, 1889).
- ³⁵ CODATA lists more than one name for some constants. It also lists the following obsolete units and non-universal constants (referencing properties characterizing the surface of the Earth instead of universal physical properties).

Angstrom star: an obsolete X-ray unit

Copper x unit: an obsolete crystallographic unit

Molybdenum x unit: another obsolete crystallographic unit

standard acceleration of gravity

standard atmosphere

standard-state pressure

CODATA's list also includes the following redundant mass ratios—the mass of each particle is already on the list.

alpha particle-electron mass ratio	muon-proton mass ratio
alpha particle-proton mass ratio	muon-tau mass ratio
deuteron-electron mass ratio	neutron-muon mass ratio
deuteron-proton mass ratio	neutron-muon mass ratio
electron-deuteron mass ratio	neutron-proton mass ratio
electron-helion mass ratio	neutron-tau mass ratio
electron-muon mass ratio	proton-electron mass ratio
electron-neutron mass ratio	proton-muon mass ratio
electron-proton mass ratio	proton-neutron mass ratio
electron-tau mass ratio	proton-tau mass ratio
electron-alpha particle mass ratio	tau-electron mass ratio
electron-triton mass ratio	tau-muon mass ratio
helion-electron mass ratio	tau-neutron mass ratio
helion-proton mass ratio	triton-electron mass ratio
muon-electron mass ratio	triton-proton mass ratio
muon-neutron mass ratio	

It also lists the following *components* of constants of Nature—not constants themselves:

Planck time	proton rms charge radius
Planck length	neutron radius
Planck charge (not on the CODATA list)	deuteron rms charge radius
Planck temperature	alpha particle rms charge radius
Planck mass	

electron charge = atomic unit of charge = elementary charge
fine-structure constant
inverse fine-structure constant
weak mixing angle
W to Z mass ratio

Finally, CODATA's 2022 listing is missing the following two constants:

Coulomb's constant (listed in the 2018 CODATA)
Planck electric impedance (we use the value listed on WolframAlpha)

³⁶ For general background on complex iteration and convergence of power towers, see, for example, John Milnor, *Dynamics in One Complex Variable*, 3rd ed. (Princeton University Press, 2006), especially Chapter 1; and Heinz-Otto Peitgen, Hartmut Jürgens, and Dietmar Saupe, *Chaos and Fractals: New Frontiers of Science*, 2nd ed. (Springer, 2004). These works discuss iterative complex maps and fixed-point structure in the complex plane.

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- ³⁷ This particular quartic and its four roots $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3, \mathfrak{K}_4$ are introduced here as part of the present proposal; I am not aware of any prior treatment of this exact “hyperbolic partition equation” in the literature. Standard tools for analyzing quartic equations and their root structure go back to Lodovico Ferrari’s 16th-century solution; a modern account is given in, e.g., Israel M. Gelfand, Mark Saul, and Andrei Shen, *The Seven-Leaf Clover and Other Problems of Mathematics and Natural Science* (Birkhäuser, 1993), Appendix on quartic equations.
- ³⁸ See Peter J. Mohr, David B. Newell, Barry N. Taylor, and Eite Tiesinga, “CODATA Recommended Values of the Fundamental Physical Constants: 2022,” *Journal of Physical and Chemical Reference Data* 52, 033104 (2023).
- ³⁹ For the formal definitions of the SI base units—second, meter, kilogram, ampere/coulomb, and kelvin—see the International Bureau of Weights and Measures (BIPM), *The International System of Units (SI)*, 9th ed. (2019).
- ⁴⁰ Roger Penrose, *Mathematics & What Exists, Closer To Truth*, Episode 2210 (2018), available at: https://www.youtube.com/watch?v=Bl7xvKY_kml. In this interview, Penrose notes that Newton’s gravitational theory is accurate to roughly one part in 10^7 , and that general relativity improves this precision to about one part in 10^{14} .
- ⁴¹ For background on natural units and Planck units as a foundational natural-unit system, see Lev D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon, 1975), §1; and Frank Wilczek, “Natural Units,” *Physics Today* 58, no. 10 (2005): 12–13.
- ⁴² CODATA, *International System of Units (SI) – Planck Unit Definitions and Uncertainty Propagation*, 2018/2022 adjustment.
- ⁴³ Richard Feynman, *QED: The Strange Theory of Light and Matter*, Princeton University Press (1985), pp. 129–130.
- ⁴⁴ The quotation is widely attributed to Pauli in physics literature, although no primary written source is known. See: A. Pais, *Niels Bohr’s Times*, Oxford University Press (1991), p. 425, which notes the attribution but indicates the lack of a documented original.
- ⁴⁵ C. Wyler, “Les constantes de la nature,” *Comptes Rendus Acad. Sci. Paris* **271** (1970): 186–188. Wyler’s formula has been discussed as an early numeric approximation to α , though without an accepted physical derivation.
- ⁴⁶ See Hans de Vries’ original post at: <https://www.physicsforums.com/threads/all-the-lepton-masses-from-g-pi-e.46055/>
- ⁴⁷ Hans de Vries originally wrote (ibid) this equation as follows:

$$\alpha^{-1/2} + \left(1 + \frac{\alpha}{2\pi}\right) \alpha^{1/2} = e^{\pi^2/4}$$

⁴⁸ Weyl, Hermann. *Symmetry*. Princeton University Press, 1952. This sentence appears in Chapter I, where Weyl is laying out his philosophical view that objectivity in science is defined by invariance under a group of transformations.

⁴⁹ Viète introduced the systematic relation between coefficients of polynomial equations and symmetric functions of their roots in ‘In Artem Analyticen Isagoge’ (1591). Modern notation differs substantially from Viète’s original symbolic system.

⁵⁰ For resolvent cubics and quartic decompositions, see any standard treatment of Galois theory, e.g. J.-P. Escofier, ‘Galois Theory’, chapter 7, or classical expositions following Ferrari’s method.

⁵¹ The invariants I and J of a binary quartic go back to Cayley and Sylvester’s 19th-century invariant theory. Their relation to the discriminant and the elliptic jjj -invariant follows the standard dictionary of algebraic geometry. See e.g. Silverman, ‘The Arithmetic of Elliptic Curves’, §3.1.

⁵² Varying a , we discover that there is a unique critical value a_* where the quartic $T(x; a)$ develops a double root. This double root occurs at an x -value x_* where the polynomial and its derivative are both zero.

$$T(x_*; a_*) = 0, \quad T'(x_*; a_*) = 0$$

At this point the discriminant vanishes, two eigenvalues of the companion matrix merge, and the system changes phase.

$$a_*^2 = 2b_* \left(\frac{8}{4\pi} \right)^2 \left(2b_* + 2 \left(\frac{4\pi}{8} \right) \right)^2$$

$$b_* = - \left(\frac{4\pi}{24} \right) + \sqrt{\left(\frac{4\pi}{24} \right)^2 + \left(\frac{4\pi}{24} \right)}$$

⁵³ Historically, this identity appears in Brahmagupta’s 7th-century work, was rediscovered by Fibonacci, and in modern terms corresponds to the multiplicativity of norms in \mathbb{C} .

⁵⁴ Euler’s four-square identity appears in his 1748 work on sums of squares and underlies the norm-preserving multiplication in Hamilton’s quaternions. The eight-square identity was later independently discovered by Degen, Graves, and Cayley, and forms the algebraic basis for the octonions. See Baez, “The Octonions,” *Bull. AMS* (2002).

⁵⁵ Hurwitz proved in 1898 that the only finite-dimensional normed division algebras over \mathbb{R} with a quadratic multiplicative norm are \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} .

⁵⁶ The cross-ratio λ is the fundamental Möbius invariant: any fractional linear transformation preserves

$$\lambda = \frac{(\kappa_1 - \kappa_3)(\kappa_2 - \kappa_4)}{(\kappa_1 - \kappa_4)(\kappa_2 - \kappa_3)}, \quad |\lambda| = 1.$$

See Beardon, *The Geometry of Discrete Groups*, chapter 2. A cross-ratio of unit modulus indicates that the four points lie on a circle and are invariant under Möbius transformations (up to orientation): a standard fact in complex projective geometry.

⁵⁷ These Newton identities (also called Newton–Girard formulas) relate the power sums S_k to the elementary symmetric polynomials. See any modern algebra text, e.g. Lang, *Algebra*, Chapter VI.

⁵⁸ Any quartic $Y^2 = T(x)$ in Weierstrass form defines an elliptic curve when $\Delta_x \neq 0$; the classical j -invariant is computed from the binary quartic invariants I, J via the standard transformation. See Silverman, op. cit.

⁵⁹ These plots are as follows:

Planck time:

https://www.wolframalpha.com/input?i=plot+pi*%28sinh%281%2F4*1%2F%28x%2Biy%29%29%29%5E2*%28e%5E5.39125836832313%2F44%29%2C+%7Bx%2C-1%2C1%7D

Planck length:

https://www.wolframalpha.com/input?i=plot+%28sinh%28sinh%281%2F7*%28x%2Biy%29%29%29%5E%28-1%29*%28e%5E1.61625918175645%2F35%29%2C+%7Bx%2C-1%2C1%7D

Planck charge:

https://www.wolframalpha.com/input?i=plot+5%2F7%5E%281%2F2%29*%283%5E%28-1%2F3%29%2F%282%5E%285%2F4%29*pi%5E%281%2F2%29*e%5E%284pi%2F32%29%2F%28gamma%281%2F4*%28x%2Biy%29%29%29%5E2%29%29*e%5E1.87554596713962%2F18%2C+%7Bx%2C-1%2C1%7D

Planck temperature:

https://www.wolframalpha.com/input?i=plot+2*%285%2F7%5E%281%2F2%29%29%5E2*%28cos%285i%2F2*1%2F%28x%2Biy%29%29%29%5E2*%28cos%287%2F5*1%2F%28x%2Biy%29%29%29%5E2*%28e%5E1.4167869859079%2F32%29%2C+%7Bx%2C-7%2C7%7D

Planck mass:

https://www.wolframalpha.com/input?i=plot+5*4pi%2F2*%28cos%287%2F5*1%2F%28x%2Biy%29%29%29%5E2*%28e%5E2.17642683817579%2F8%29%2C+%7Bx%2C-8%2C8%7D

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- ⁶⁰ John Archibald Wheeler, “*Information, Physics, Quantum: The Search for Links*,” in *Complexity, Entropy, and the Physics of Information*, ed. Wojciech H. Zurek (Addison-Wesley, 1990), p. 3.
- ⁶¹ For background on $\zeta(2)$, $\zeta(3)$, and the role of polylogarithmic constants in mathematical physics, see: Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976—chapters on the Riemann zeta function; and Don Zagier, “Values of Zeta Functions and Their Applications,” in *First European Congress of Mathematics*, 1994.
- ⁶² For geometric background on the figure-eight knot complement, its ideal tetrahedral decomposition, and hyperbolic volume, see: William Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton lecture notes, 1978–1981—Chapter 4 covers hyperbolic structures on knot complements; where he originally identifies the figure-eight complement as simplest hyperbolic 3-manifold; and Colin Adams, *The Knot Book*, American Mathematical Society, 2004.
- ⁶³ For the relationship between hyperbolic 3-manifold volumes and dilogarithms, see: Walter Neumann & Don Zagier, “Volumes of Hyperbolic Three-Manifolds,” *Topology* 24 (1985): 307–332; and Leonard Lewin, *Polylogarithms and Associated Functions*, North-Holland, 1981.
- ⁶⁴ See: Don Zagier, “The Dilogarithm Function,” in *Frontiers in Number Theory, Physics, and Geometry II*, Springer (2007), pp. 3–65; and L. Lewin (1981), *ibid.*
- ⁶⁵ For the classification of the figure-eight complement as the minimal-volume orientable cusped hyperbolic 3-manifold, see: Colin Adams, “The Noncompact Hyperbolic 3-Manifold of Smallest Volume,” *Proc. Amer. Math. Soc.* 100 (1987): 601–606; and Cao & Meyerhoff, “The Orientable Cusped Hyperbolic 3-Manifolds of Minimal Volume,” *Invent. Math.* 146 (2001): 451–478.
- ⁶⁶ For the derivation of the Cayley–Menger determinant and the $1/288$ normalization for tetrahedral volume, see H. S. M. Coxeter, *Introduction to Geometry*, Wiley, 1961 — simplex volume formulas; Gram matrices; and J. B. Robins, “The Cayley–Menger Determinant,” *Mathematics Magazine* 66 (1993): 38–43.
- ⁶⁷ See: www.chip-architect.com/news/2004_10_04_The_Electro_Magnetic_coupling_constant.html
- ⁶⁸ Albert Einstein, “Autobiographical Notes,” in *Albert Einstein: Philosopher–Scientist*, edited by Paul Arthur Schilpp, Library of Living Philosophers, Vol. VII (Open Court, 1949).
- ⁶⁹ For standard treatments, see William P. Thurston, *The Geometry and Topology of Three-Manifolds* (Princeton lecture notes, 1979), chapter 3; Colin Adams, *The Knot Book* (W. H. Freeman, 1994), chapter 9; and Jeffrey Weeks, *The*

Shape of Space (CRC Press, 2002). These discuss the hyperbolic structure on the figure-eight knot complement and its ideal tetrahedral decomposition.

⁷⁰ Sharply characterizing the partition structure of this minimal system requires that we understand the zeros (bounds) it puts in place. Under power operations of the imaginary golden ratio, this dilogarithmic structure maintains a 6-cycle composed of 2 mirrored 3-cycles. This 6-cycle contains 2 zeros.

$$i \left[\operatorname{Li}_2 \left(\frac{1}{\varphi_i^1} \right) - \operatorname{Li}_2(\varphi_i^1) \right] = V_{fe}$$

$$i \left[\operatorname{Li}_2 \left(\frac{1}{\varphi_i^2} \right) - \operatorname{Li}_2(\varphi_i^2) \right] = \frac{2}{3} V_{fe}$$

$$i \left[\operatorname{Li}_2 \left(\frac{1}{\varphi_i^3} \right) - \operatorname{Li}_2(\varphi_i^3) \right] = 0$$

$$i \left[\operatorname{Li}_2 \left(\frac{1}{\varphi_i^4} \right) - \operatorname{Li}_2(\varphi_i^4) \right] = -\frac{2}{3} V_{fe}$$

$$i \left[\operatorname{Li}_2 \left(\frac{1}{\varphi_i^5} \right) - \operatorname{Li}_2(\varphi_i^5) \right] = -V_{fe}$$

$$i \left[\operatorname{Li}_2 \left(\frac{1}{\varphi_i^6} \right) - \operatorname{Li}_2(\varphi_i^6) \right] = 0$$

⁷¹ For the classification of even unimodular lattices and the uniqueness and properties of the Leech lattice (including the absence of norm-2 vectors), see J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed. (Springer, 1999), chapters. 4, 10–11.

⁷² The volume of the 24-dimensional unit ball is $V_{24} = \pi^{12}/12!$; for derivations and tables see, e.g., Conway & Sloane (1999), Appendix 2. The optimality of the Leech lattice packing and its matching of the Cohn–Elkies bound were proved by Henry Cohn, Abhinav Kumar, Stephen Miller, Danylo Radchenko, and Maryna Viazovska, ‘The sphere packing problem in dimension 24,’ *Annals of Mathematics* 185 (2017): 1017–1033.

⁷³ This lattice can be constructed by interweaving three copies of the E_8 lattice, stitched together by the binary Golay code, the unique error-correcting code $E_8 \oplus E_8 \oplus E_8$. See, On the construction of the Leech lattice from three E_8 lattices via the binary Golay code, J. H. Conway, ‘A simple construction for the Leech lattice,’ *Inventiones Mathematicae* 76 (1984): 507–513; and Conway & Sloane (1999), chapter 10.

A useful way to express how “three E_8 -like layers” stitch together into a 24-dimensional transform domain is through a stitching recipe:

$$s_r \delta_8^3 = s_r \rho_{8D}^3 = V_{24} = \frac{\pi^{12}}{12!} \quad s_r = \frac{2^{14}}{44 \cdot 35 \cdot 18 \cdot 5}$$

$$\delta_8 = \left(\frac{4\pi}{\Gamma(5)} \right) \left(\frac{4\pi}{16} \right)^3 \quad \rho_{8D} = \left(\frac{4\pi}{\Gamma(5)} \right) \left(\frac{4\pi}{16} \right)^3$$

Here s_r = the stitching recipe, δ_8 = the densest hypersphere lattice packing in 8 dimensions, ρ_{8D} = the maximum-density sphere packing in 8 dimensions, both of which are equal to each other, and define the density of the E_8 lattice.

This stitching recipe identifies the combinatorial overhead required to interlace three 8-dimensional layers into a single 24-dimensional coherent whole.

The factor 2^{14} reflects an underlying binary structure of 14 layers, while divisors 44, 35, and 18—correspond to the combinatorial partitions associated with t_p , l_p , and q_p .

⁷⁴ The classification of positive-definite odd unimodular lattices in dimension 23, including the 117 rooted classes and the unique rootless class (the shorter Leech lattice), is summarized in Conway & Sloane (1999), chapter 16, especially Tables 16.7–16.11; see also Gabriele Nebe and Neil Sloane, ‘A Catalogue of Lattices,’ online tables.

⁷⁵ The external geometric actions also include composite units, the components of which are already included in our list.

ampere = A = C/s	ohm = Ω = m ² kg/s C ²
farad = F = s ² C ² /m ² kg	pascal = Pa = kg/s ² m
henry = H = m ² kg/C ²	siemens = S = sC ² /m ² kg
hertz = Hz = 1/s	tesla = T = kg/sC
joule = m ² kg/s ²	volt = V = m ² kg/s ² C
newton = N = m kg/s ²	watt = W = m ² kg/s ³
noether = \mathcal{N} = m kg/s	weber = Wb = m ² kg/sC

⁷⁶ <https://mathworld.wolfram.com/FigureEightKnot.html>

⁷⁷ https://www.wolframalpha.com/input?i=Li_2%28x%29

⁷⁸ https://www.wolframalpha.com/input?i=i*%5BLi_2%281%2Fi%29-Li_2%28i%29%5D

⁷⁹ These two division structures—one “cosine-like” and one “sine-like”—characterize neutron-electron and neutron-proton radii ratios.⁷⁹

$$\frac{r_n}{r_e} = \left(\frac{4\pi}{\Gamma(5)} \right)^2 = \sum_{n=1}^{\infty} \frac{\cos\left(n \frac{\pi}{3}\right)}{n^2} \quad \frac{r_n}{r_+} = K = \sum_{n=1}^{\infty} \frac{\sin\left(n \frac{\pi}{2}\right)}{n^2}$$

$$r_+ = 8.43431614404474 \dots \times 10^{-16} \text{ m} \quad \text{prediction}$$

$$r_+ = 8.4075(64) \times 10^{-16} \text{ m} \quad \text{CODATA 2022, } \sigma = +4.18$$

$$r_+ = 8.414(19) \times 10^{-16} \text{ m} \quad \text{CODATA 2018, } \sigma = +1.05$$

$$r_n = 7.72554339835845 \dots \times 10^{-16} \text{ m} \quad \text{prediction}$$

$$r_n = 8 \times 10^{-16} \text{ m} \quad \text{Povh, B.: Rith, K.: Scholz, C.: Zetsche, F. (2002)}$$

Where r_n = the neutron radius, r_e = the classical electron radius, r_+ = the proton radius, π = Archimedes' constant, $\Gamma(x)$ = the gamma function, and K = Catalan's constant.

⁸⁰ [https://en.wikipedia.org/wiki/E_\(mathematical_constant\)](https://en.wikipedia.org/wiki/E_(mathematical_constant))

⁸¹ <https://mathworld.wolfram.com/DominoTiling.html>

⁸² <https://www.wolframalpha.com/input?i=1.7916228120695934247305470>

⁸³ <https://mathworld.wolfram.com/LemniscateConstant.html>

⁸⁴ <https://mathworld.wolfram.com/GaussConstant.html>

⁸⁵ <https://mathworld.wolfram.com/UbiquitousConstant.html>

⁸⁶ <https://mathworld.wolfram.com/SierpinskiConstant.html>

⁸⁷ https://en.wikipedia.org/wiki/Universal_parabolic_constant

⁸⁸ <https://mathworld.wolfram.com/Omega-2Constant.html>

⁸⁹ <https://mathworld.wolfram.com/Euler-MascheroniConstant.html>

⁹⁰ https://en.wikipedia.org/wiki/Dottie_number

⁹¹ <https://mathworld.wolfram.com/HyperbolicCotangent.html>

⁹² <https://mathworld.wolfram.com/LaplaceLimit.html>

⁹³ <https://mathworld.wolfram.com/GoldenRatio.html>

⁹⁴ <https://mathforums.com/t/imaginary-golden-ratio.17605/>

⁹⁵ <https://mathworld.wolfram.com/PlasticConstant.html>

⁹⁶ <https://mathworld.wolfram.com/RamanujanContinuedFractions.html>

⁹⁷ <https://mathworld.wolfram.com/BesselFunctionZeros.html>

⁹⁸ https://en.wikipedia.org/wiki/Feigenbaum_constants

⁹⁹ <https://mathworld.wolfram.com/SilverConstant.html>

¹⁰⁰ <https://mathworld.wolfram.com/Hafner-Sarnak-McCurleyConstant.html>

¹⁰¹ The QRS constant C_Q is the universal critical threshold for phase locking in nonlinearly coupled oscillators, originally derived by Pikovsky, Rosenblum & Kurths in their study of global synchronization. See A. Pikovsky, M. Rosenblum & J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, 2001).

Both the QRS constant C_Q , and the associated QRS constant C_{QA} , characterize structural limits of the Riemann zeta function:

$$C_Q = \text{root of } \zeta\left(\frac{1}{2}, \frac{x}{2}\right), \quad 4C_{QA} = \text{root of } \zeta\left(\frac{3}{2}, \frac{C_Q}{2}\right).$$

¹⁰² <https://mathworld.wolfram.com/QRSCONSTANT.html>

¹⁰³ See Cayley, “*A Memoir on the Theory of Matrices*” (1858), and Sylvester, “*On the Theory of Linear Transformations*” (1851), the foundational works introducing matrices as algebraic objects.

¹⁰⁴ The geometric interpretation of the determinant as volume scaling is standard; see Sheldon Axler, *Linear Algebra Done Right* (Springer), Chapter 7.

¹⁰⁵ For unimodular lattices and their automorphism groups, John Conway & Neil Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed. (1999), Chapters 2–3.

¹⁰⁶ For the spectral theorem and physical role of Hermitian operators, see Reed & Simon, *Methods of Modern Mathematical Physics*, Vol. I.

¹⁰⁷ The variational formulation of mechanics originates in Lagrange’s *Mécanique Analytique* (1788).

¹⁰⁸ Hamilton’s reformulation of mechanics is developed in his “On a General Method in Dynamics” (1834–35); see also José & Saletan, *Classical Dynamics*.

¹⁰⁹ William Rowan Hamilton, *Lectures on Quaternions* (1853), Lecture 1.

¹¹⁰ Dirac introduced this correspondence principle in *The Principles of Quantum Mechanics* (1930).

¹¹¹ Matrix mechanics originated with Heisenberg’s 1925 paper “Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen,” *Z. Phys.*, and was formalized by Born and Jordan in “Zur Quantenmechanik,” *Z. Phys.* (1925), and Born–Heisenberg–Jordan in “Zur Quantenmechanik II,” *Z. Phys.* (1926).

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- ¹¹² For the relation between Lie algebras and exponential maps generating continuous symmetries, see Hall, *Lie Groups, Lie Algebras, and Representations* (2003).
- ¹¹³ The exponential of the shift matrix and its connection to Pascal’s triangle is discussed in Graham, Knuth & Patashnik, *Concrete Mathematics*, Chapter 7.
- ¹¹⁴ The Klein–Gordon equation was developed by Klein and Gordon in 1926, and Dirac introduced his linear relativistic wave equation in “The Quantum Theory of the Electron,” (1928) *Proc. Roy. Soc. A*.
- ¹¹⁵ For the construction and spectral properties of companion matrices, see Horn & Johnson, *Matrix Analysis*, Chapter 3.
- ¹¹⁶ The resolvent cubic and quadratic factorization for quartics go back to Ferrari’s 16th-century method; see van der Waerden, *Modern Algebra*, Vol. 1, §§10–11.
- ¹¹⁷ The use of adjoint operators and two-field formulations to restore reality and time-reversal symmetry is standard in field theory; for Noether’s theorem and U(1) phase invariance, see e.g. Weinberg, *The Quantum Theory of Fields*, Vol. 1, §7.5.
- ¹¹⁸ The reduction of an n th-order scalar ODE to a first-order system in \mathbb{R}^n is standard; see Coddington & Levinson, *Theory of Ordinary Differential Equations*, Chapter 1.
- ¹¹⁹ For Yukawa-type couplings mixing components of multiplets in Dirac theories, see Peskin & Schroeder, *An Introduction to Quantum Field Theory*, Chapter 11.
- ¹²⁰ Dirac’s factorization of the Klein–Gordon operator is given in his 1928 paper, “The Quantum Theory of the Electron,” *Proc. Roy. Soc. A*.
- ¹²¹ For the role of Clifford algebras and spinors in relativistic quantum theory, see Penrose & Rindler, *Spinors and Space-Time*, Vol. 1.
- ¹²² On the emergence of writing as an externalization of memory, see Denise Schmandt-Besserat, *How Writing Came About*, University of Texas Press (1996).
- ¹²³ Euclid’s *Elements* is the classical source for the first complete deductive geometric system; see Euclid (trans. Heath), Books I–XIII.
- ¹²⁴ For the historical development of algebra and al-Khwarizmi’s foundational role, see Berggren, *Episodes in the Mathematics of Medieval Islam*.
- ¹²⁵ For historical accounts of the independent development of calculus by Newton and Leibniz, see Boyer & Merzbach, *A History of Mathematics*, Chapter 12.
- ¹²⁶ Matrices as algebraic entities were introduced by Cayley in his 1858 memoir; see Cayley, *Phil. Trans. Roy. Soc.*, 1858.

¹²⁷ For the formalization of information as a measurable quantity, see Shannon, “A Mathematical Theory of Communication,” *Bell Syst. Tech. J.*, 1948.

¹²⁸ The figure-eight knot complement is the simplest nontrivial hyperbolic 3-manifold; see Thurston, *Geometry and Topology of Three-Manifolds*, Chapter 4.